A COURSE IN EXTERIOR BALLISTICS

ORDNANCE TEXTBOOK

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COURSE IN EXTERIOR BALLISTICS

ERRATA

P. 5, line 4 from bottom; I should be 1.
P. 11, line 11 from bottom of text; before "or see," insert footnote reference, 2.
P. 11, bottom; add footnote 2: "See supplement F."
P. 28, line 9; Integral should have lower limit, t_n.
P. 30, prob. 25; x_2 should be x^2.
P. 39, line 11 from bottom; delete "the."
P. 39, line 5 from bottom; delete "for each arc."
P. 39, line 4 from bottom; delete "difference for each arc, and hence the correct range."
P. 44, line 20; after "theoretically," add footnote reference, 3.
P. 44, bottom; add footnote 3: "See p. 63."
P. 50, footnote, last eq.; Parenthesis should read: $(y''_t + \frac{\Delta y''}{2})$.
P. 54, line 13; for "a function of," read "dependent on."
P. 57, line 17; for "equations 47," read "equation 47."
P. 59, line 7 from bottom; for "book," read "chapter."
P. 59, line 6 from bottom; for "some in Chapter XI," read "that for site."
P. 60, line 5 from bottom; for "$\Delta\phi$" read "$\delta\phi$."
P. 62, line 19; instead of "whence," the line should read: "whence, by theorem 9, page 21:"
P. 63, line 9 from bottom; add footnote reference, 2.
P. 63, line 4 from bottom; add footnote reference, 3.
P. 63, bottom; add footnotes: "[2] See p. 44," and "[3] See Problem 55, p. 69, post."
P. 64, line 3; first word should be "in."

P. 67, last line; to read: "radius of the earth, and ω was the angle of fall."

P. 69, lines 1, 2; change "x - X" to "X - x" in each line.

P. 69, lines 12, 13; change "Δs" to "δs" in each line.

P. 70, fig. 6; the "J" which marks the angle should be "j."

P. 71, first eq.; "(E + J)" should be "(E + j)."

P. 75, eq. 79; delete the comma in the 1019.

P. 81, eq. 94; $(x'\rho - y'\nu)$" should be "$(\nu y' - \rho x')$."

P. 81, eq. 95; change "cos α" to "sin α" in first eq. only.

P. 84, 85; column to right of the T column should have had all horizontal lines broken, to indicate that several columns have been omitted.

P. 85, No. 56; Integral should have upper limit, "T."

P. 88, first eq.; "μ_t" should be "μ_{t_0}."

P. 88, values of A, B, C; lower limits on integrals are 0.

P. 88, value of A; "$(x'\rho - y'\nu)$" should be "$(\nu y' - \rho x')$."

P. 88, line 7 from bottom; integral should have upper limit, "T"

P. 93, fig. 13; reference should be to fig. 12, instead of to fig. 1.

P. 95, footnote; the word "second" should not be italicized.

P. 97, line 3 from bottom of text; delete the footnote reference.

P.113, line 1; line should read: "Substituting from equations 117 and 118 in equations 116;"

P.114, last 5 lines; change "dx" to "dX," four times.

P.117, footnote; the first Δ in the equation should have for subscript, "t - 1."

P.126, col. 2; "tΔ" should be "t_Δ"

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WAR DEPARTMENT,
WASHINGTON, *December 14, 1920.*

The following publication, entitled "A Course in Exterior Ballistics," is prepared for the information and guidance of all concerned.

[062.1, A. G. O.]

BY ORDER OF THE SECRETARY OF WAR:

PEYTON C. MARCH,
Major General, Chief of Staff.

OFFICIAL:

P. C. HARRIS,
The Adjutant General.

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EXTERIOR BALLISTICS.

INTRODUCTION.

The work of the ballistic computer is divided into three parts: (1) the computation of the elements of standard trajectories; (2) the computation of differential corrections, whereby the elements of a standard trajectory may be corrected for nonstandard conditions; and (3) the utilization of the foregoing to construct range tables from firing records.

As is shown in Chapter VI, the World War ushered in a new era in the handling of ballistic problems. This may be called the period of numerical integration.

The approximations, which had been used to modify the simple Newtonian equations of motion into such form that they could be formally integrated, gave place to precise numerical integration of these equations in their original form.

Practically parallel progress was made in all of the allied countries during the war.

In America the great step in the computation of trajectories was the introduction of numerical integration. Numerical integration had long been used in astronomical calculations, and so it was natural that an astronomer and mathematician, Prof. F. R. Moulton, of the University of Chicago, while serving as a major in the Ordnance Department of the United States Army, should have applied this method to ballistics. The practical work of this method was materially reduced by formulas for "integrating ahead" later introduced.

The work can be still further reduced by a variant of Maj. Moulton's method, known as the tangent-reciprocal method. But this method has the pedagogical drawback of obscuring the physical meaning of the steps involved, and hence will not be given first place in this book.[1]

Another improvement is the change in the analytic interpretation of the equations of motion. Formerly the x-axis was conceived of as tangent to the earth at the gun, the system being Cartesian. In the modern conception, the x of a point is measured along the curved surface of the earth, and the y is measured vertically from this surface.

The progress in the computation of the differential corrections has involved more steps. Differential equations for the corrections were devised at the same time that numerical integration was introduced.

[1] See Supplement A.

But these had the difficulty of requiring an independent computation for each correction.

This difficulty was removed by the discovery at Aberdeen of a method of solution by means of a set of adjoint equations involving several auxiliary variables. All the differential corrections can be expressed in terms of these auxiliary variables. A physical derivation of these variables, and of the corrections based upon them, was later found. Matters were further simplified by reducing all of these variables to expressions in terms of one variable and its derivatives. A physical derivation of this step was at once forthcoming.

One further step should be noted, namely the development of the weighting-factor curves for zero elevation, which have been of great value in interpolation.

The present method of constructing range tables out of firing records is a logical result of substituting for the Ingalls tables the new methods of computation. Tables, to take the place of the computations now necessary, are now being constructed by the technical staff at Washington.

The credit for the above-described development is largely due to Mr. J. J. Arnaud, Master Computer, Ordnance Department; Prof. A. A. Bennett, of the University of Texas, then Captain, Ordnance Department, U. S. Army; Prof. G. A. Bliss, of the University of Chicago, Technical Expert, Aberdeen; Mr. Philip Franklin, Computer, Aberdeen; Dr. T. H. Gronwall, Mathematics and Dynamics Expert, Technical Staff; Prof. H. H. Mitchell, of the University of Pennsylvania, Master Computer, who organized the range table computation work at Aberdeen; Dr. J. F. Ritt, of Columbia University, Master Computer, Technical Staff, and their associates, in addition to those mentioned elsewhere herein.

Most of the written material on the ballistic progress made during the war consists of scattered blue prints, some printed at Aberdeen and some at Washington. These pamphlets overlap in spots, contain some hiatuses, and do not agree in symbology and nomenclature. Through a three-cornered correspondence between the Technical Staff, Ordnance Office, (War Department), at Washington, D. C., and the Ordnance School of Application and the Ballistic Section, Aberdeen Proving Ground, Md., a uniform symbology and nomenclature have been established as standard.

The first course of instruction in these new ballistic methods ever given in this country was given at the Ordnance School of Application in the winter of 1919–20 by Capt. Roger Sherman Hoar, Coast Artillery, then in charge of the Ballistic Section of the Proof Department at Aberdeen. This present book is based upon the papers used in that course, and uses the standard symbology and nomenclature established as above.

It is assumed that the student is thoroughly grounded in algebra and plane trigonometry, and knows enough calculus to appreciate the meaning of a derivative, a differential, and a definite integral. On that basis, this book gives, in Chapters I to IV, the irreducible minimum of higher mathematics necessary to understand all points involved in the later chapters.

The book then takes up in succession: An introduction to modern ballistic methods (Chaps. V and VI); the computation of trajectories (Chaps. VII and VIII); the computation of differential corrections (Chaps. IX to XV); and the construction of range tables (Chap. XVI). Alternative methods, elaborations of certain points, and a brief mention of the more involved mathematical processes necessary to the computation of antiaircraft range tables are reserved for supplements.

Each chapter is followed by a series of questions, designed to bring out the salient features of the chapter. The answers to most of these questions will be found categorically stated in the text, but some under each chapter will require a small degree of original thought on the part of the student.

Throughout the book the attempt is made to explain as much as possible from the viewpoint of physics rather than from the viewpoint of abstract mathematics.

CHAPTER I.

PARTIAL DIFFERENTIATION.

Before defining the "partial differentiation" of a function, let us define the word "function." u is called a "function" of x, y, z, etc. if, when x, y, z, etc., are given, the value of u is determined. Note the broadness of this definition. Thus $u=xy$ can be regarded as a function of x, y, *and* z, although it is evident that u is not in the least dependent on z.

The functional relation is expressed in the general form:

$$u=f(x, y, z, \quad . \quad . \quad . \quad).$$

One should be sure to notice the fundamental fact that this expression, as it stands, does not take into consideration any relationship which may exist between any of the variables in the parenthesis; in other words, these variables may or may not be independent *in that expression.*

Usually the functional equation can be altered so as to make x explicitly a function of u, y, z, etc.; y explicitly a function of u, x, z, etc. In case such a conversion is either impossible or even merely inconvenient, it is better to regard the equation, in its original form, as defining u as a function of x, y, z, etc., or x as a function of u, y, z, etc.; and to differentiate it as it stands, using the differential method. Such an equation is that in problem 3, to follow. Thus, in an equation containing n variables, any one of these variables can generally be regarded as dependent, and the remaining $n-1$ as independent.

We are now in a position to define the term "partial derivative." If $u=f(x, y, z \quad . \quad . \quad . \quad)$, then the partial derivative of u with respect to x is obtained by treating as constants all the other variables in the parenthesis, and differentiating with respect to x, in the ordinary manner.

But the *partial* derivative is written $\frac{\partial u}{\partial x}$, instead of $\frac{du}{dx}$, so as to indicate that x is *one of several* independent variables, instead of being the *sole* independent variable. Thus $\frac{\partial u}{\partial x}$ means the rate of change of u with respect to a change in x alone out of several independent variables.

An important point to note in this connection is that the symbol $\frac{\partial u}{\partial x}$ is highly ambiguous, i. e., it has entirely different meanings, and its partial derivatives have entirely different values, according to what

are regarded as the independent variables. To avoid this ambiguity, the subscript notation should be used. Thus $\frac{\partial u_{xyz}}{\partial x}$ means that u should be expressed as a function of x, y, and z, and then differentiated with respect to x: $\frac{\partial u_{XY}}{\partial x}$ means that u should be expressed as a function of X and Y before differentiating, etc.[1]

In practice, the subscripts may be omitted whenever it is self-evident, from the problem, just what are to be treated as the independent variables in each differentiation.

PROBLEMS.

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ and $\frac{\partial y}{\partial x}$ in each of the following:

(1) $z=x \log y$.

(2) $z=ax^2+2bxy-cy^2$.

(3) $z=-\frac{e^{xy}}{x^2+y^2}$.

(4) $x^2+y^2+z^2=a^2$.

All of the theorems of partial differentiation can be derived as special cases of the following *general theorem*:

1. *If u is a function of X, Y, Z $\cdots$, each of these in turn being a function of x, y, z $\cdots$, then (with certain assumptions as to continuity):*

$$\frac{\partial u_{xyz\ldots}}{\partial x}=\frac{\partial u_{XYZ\ldots}}{\partial X}\cdot\frac{\partial X_{xyz\ldots}}{\partial x}+\frac{\partial u_{XYZ\ldots}}{\partial Y}\cdot\frac{\partial Y_{xyz\ldots}}{\partial x}+\cdots$$

and similarly for $\frac{\partial u_{xyz\ldots}}{\partial y}$, *etc.*

This theorem can be derived by the use of undetermined coefficients, or see Osgood, p. 296.

This theorem is called "general," not in the sense that it is the fundamental basis of the other theorems, for many of them have a simpler derivation, and some, in fact, may serve as steps in the derivation of theorem 1. Nor is it advisable to use theorem 1 when some simpler formula is available. But, as theorem 1 has a form easy to remember, and as any other formula *can* be derived as a special case thereof, it serves as a good memory peg on which to hang the whole subject of partial differentiation.

The student should particularly note at this juncture that, although any partial derivative may be obtained by the differential method;

[1] For further exposition of the subscript notation, see Osgood, *Differential and Integral Calculus*, p. 306.

yet, once formed, its numerator and denominator are inseparable and neither can be canceled. Witness the absurd results which would follow from performing all possible cancellations in theorem 1.

The following general principles may be used in deriving special formulas from theorem 1:

(*a*) Whenever in partial differentiation any given variable is regarded as dependent on one independent variable alone, then, in the expression for the derivative of the former with respect to the latter, the operator ∂ should be changed to the operator d.

(*b*) The derivative of a variable, with respect to another variable of which the first is regarded as independent, is zero.

(*c*) The derivative of a variable with respect to itself is unity.

(*d*) If, for all values of its variables, a given function is *explicitly* a constant, then the derivative of that function with respect to any of such variables is zero.

The following special formulas may be derived from theorem 1:

2. $$du=\frac{\partial u}{\partial X}dX+\frac{\partial u}{\partial Y}dY+\frac{\partial u}{\partial Z}dZ\cdot\cdot$$

This is called the expression for the "total differential" of u.

3. *If u be a particular function of X and Y, namely XY, then:*

$$\frac{\partial u}{\partial x}=Y\frac{\partial X}{\partial x}+X\frac{\partial Y}{\partial x},$$

which is the analog of the following formula of total differentiation:

$$du=YdX+XdY.$$

4. *If u equals $\frac{XY}{Z}$, then:*

$$\frac{\partial u}{\partial x}=\frac{Y}{Z}\frac{\partial X}{\partial x}+\frac{X}{Z}\frac{\partial Y}{\partial x}-\frac{u}{Z}\frac{\partial Z}{\partial x}=\frac{u}{X}\frac{\partial X}{\partial x}+\frac{u}{Y}\frac{\partial Y}{\partial x}-\frac{u}{Z}\frac{\partial Z}{\partial x}.$$

5. *If u equals Yx, then:*

$$\frac{\partial u}{\partial x}=x\frac{\partial Y}{\partial x}+Y.$$

Similar special formulas can be derived for other special relations which may exist between u, X, Y, Z, etc.

PROBLEMS.

NOTE.—The following problems should be treated exactly as though the symbols y', x'', E, etc., were the simpler-looking symbols of the preceding problems and explanation. For the purposes of any given differentiation, x, x', x'', y, y', y'',

and E have no relation to each other except that given in the hypotheses. But the work on these problems should be carefully saved for later reference. In Chapter V a meaning will be given to each of these symbols, and in Chapter IX it will be seen that these problems, taken in order, constitute almost the entire proof of some important ballistic formulas.

PROBLEMS.

(5) Given that x'' and y'' are each functions of the independent variables x', y', and y, evaluate dx'' and dy'' in terms of dx', dy', and dy and some partial derivatives.

For problems 6 to 13, the following is given:

$$x'' = -Ex'$$

$$y'' = -Ey' - g$$

where g is a constant; and E is a "function" of x', y', y and x, although not dependent on x.

(6) $\frac{\partial x''}{\partial x}$. (7) $\frac{\partial y''}{\partial x}$. (8) $\frac{\partial x''}{\partial y}$. (9) $\frac{\partial y''}{\partial y}$.

(10) $\frac{\partial x''}{\partial x'}$. (11) $\frac{\partial y''}{\partial x'}$. (12) $\frac{\partial x''}{\partial y'}$. (13) $\frac{\partial y''}{\partial y'}$.

The foregoing problems can be done either by one of the special theorems or by theorem 1. It is advisable not to use theorem 1; but if it be used, care should be taken to observe that x' and y' each enter into the expressions for x'' and y'' in two capacities: i. e., as a variable of the X sort and as a variable of the x sort. To illustrate this, perform the following problem:

(14) $x'' = EZ$
$E = f(x', y', y)$
$Z = x'$

Evaluate $\frac{\partial x''}{\partial x'}$ by theorem 1. Also by theorem 5.

QUESTIONS ON CHAPTER I.

1. What is a partial derivative?
2. How, in general, is partial differentiation performed?
3. Does $\frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$ equal $\frac{\partial z}{\partial x}$? If so, why? If not, why not?
4. If $w = f(p, q, r)$, and $\frac{\partial w}{\partial r}$ is evaluated in terms of p, q, and r, and we then learn that a certain special relationship—extraneous to the equation $w = f(p, q, r)$—exists between p, q, and r, will this fact change the value of $\frac{\partial w}{\partial r}$? If the special relationship is such that w equals a constant, will that fact make $\frac{\partial w}{\partial r} = 0$? If so, why? If not, why not?
5. Explain the meaning of, and the need for, the subscript notation.

CHAPTER II.

SUCCESSIVE APPROXIMATIONS.

An equation which determines the numerical value of a quantity may generally be expressed in a variety of alternative ways. Thus the fact that x is the square-root of w may be expressed:

$$(a)\quad x = \pm\sqrt{w}$$

$$(b)\quad x - \frac{w}{x} = 0$$

$$(c)\quad \begin{cases} x = wy \\ y = \dfrac{1}{x}. \end{cases}$$

Here (a) defines x *ex*plicitly; (b) defines x *im*plicitly; and (c) consists of simultaneous equations in x and y, each defined as a function of the other.

Forms analagous to (a) are not always forthcoming. Consider, for example, the determination of a number x, such that x is equal to 500, plus 1,000 times its own trigonometric sine, x being measured in minutes of arc. There is no known explicit form for x. We have, however:

$$(b)\quad x = 500 + 1{,}000 \sin x.$$

$$(c_1)\quad \begin{cases} x = 500 + y \\ y = 1{,}000 \sin x \end{cases}$$

$$(c_2)\quad \begin{cases} x = 500\, y \\ y = 1 + 2z \\ z = \sin x \end{cases}$$

An implicit equation may frequently be replaced by a system of equations (similar in general to equations of the sort c) so chosen as to be convenient for solution by a computational procedure known as "successive approximations." This, as its name suggests, is a method of starting with an apt number, largely arbitrary, and by successive substitutions securing a sequence of approximate evaluations of x, approaching the precise value to within any desired degree of precision

For example, extract the square root of w by the following equations:

$$y=\frac{w}{z}$$

$$x=\frac{1}{2}(y+z)$$

$$z=x.$$

Expressed as a formula:

$$y_n=\frac{w}{x_n}$$

$$x_{n+1}=\frac{1}{2}(y_n+x_n)$$

Extracting the square root of 2 by this pair of equations, first assuming 2 to be approximately its own root, gives the following successive values:

n	x	y
1	2	1
2	$\frac{3}{2}$	$\frac{4}{3}$
3	$\frac{17}{12}$	$\frac{24}{17}$
4	$\frac{577}{408}$	

and so on. Expressed in ordinary language, we have the following rule for the extraction of square root: Divide the number by an approximation to the square root desired; the arithmetic mean of the divisor and quotient is a new approximation.

It can be shown that if any approximate square root checks with the preceding approximate root to n figures, then the new approximate root is correct to at least $2n-1$ figures.

Had a negative value been taken initially for x_1, the negative square root of 2 would have been approached. The separating value, zero, causes the method to fail, as one would expect.

An attempt to extract the square root of 2 by the equations:

$$y=\frac{1}{x}$$

$$x=wy$$

produces the following results:

n	x	y
1	2	$\frac{1}{2}$
2	1	1
3	2	$\frac{1}{2}$
4	1	1

and so on. A similar repetition would occur for any initial value for x, other than zero or infinity. This shows that not all sets of simultaneous equations are adapted to solution by successive approximations.

The method of successive approximations has the advantage that the accuracy of each step is independent of the accuracy of the preceding step. A mistake in any single step, therefore, while it may prolong the work, will not vitiate the final result.

PROBLEMS.

(15) Extract the square root of 100, taking 25 as the approximate square root and carrying each division to three decimal places. Continue until two successive answers check to three decimal places.

(16) Assume that 1.41459 is an approximation to the square root of 2. Perform one step of getting the root with greater precision, and give the result to only the number of places certain to be correct.

(17) Replace $x=500+1{,}000 \sin x$, by the system:

$$y=2{,}000 \sin x$$

$$z=\frac{1}{2}(1{,}000+y)$$

$$x=z$$

and use as formulas for successive approximation:

$$y_n=2{,}000 \sin x_n$$

$$x_{n+1}=\frac{1}{2}(1{,}000+y_n)$$

x is expressed in minutes of arc. Find x and y correct to the nearest unit.

(18) y is a tabular function of the time, t. The following is a tabulation showing the value of y corresponding to each of certain values of t.

t	y	a	b	c
60	1,569.4			
62	1,052.4	−517.0		
64	512.5	−539.9	−22.9	
66	−46.6	−559.1	−19.2	+3.7

a, b, and c are, respectively, the first, second, and third differences of y; i. e., the a of any line is obtained by subtracting, from the y of that line, the y of the line before; the b of any line is obtained by subtracting, from the a of that line, the a of the line before, etc.

To find the value of t corresponding to some given, non-tabular value of y:

Let y_t = the value of y at time, t;

y_0 = the value of y at the time, t_0;

and i = the tabular interval in t.

Representing by Δt the value of $\frac{t-t_0}{i}$, which is to be considered with its algebraic sign, the formula to be used is:

$$6. \qquad \Delta t = \frac{y_t - y_0}{a + \frac{1+\Delta t}{2!} b + \frac{(1+\Delta t)\,(2+\Delta t)}{3!} c}$$

where a, b, and c are the values taken *from the same line* on which t_0 and y_0 occur. The Δt so obtained must be multiplied by i, to obtain $-t_0$.

The formula is solved by successive approximation, the first approximation being

$$\Delta t = \frac{y_t - y_0}{a}$$

Find the value of t corresponding to $y_t = 0$, using $t_0 = 64$. Check by using $t_0 = 66$.

This formula, obtainable from the result of problem 30, page 31, may be used to interpolate in either direction from any tabulated values, but it *requires* the use of "*receding* differences;" i. e., differences that, in the method of writing used above, occur on the *same* line with t_0 and y_0.[1]

At the close of Chapter IV problems will be given involving a combination of numerical integration and successive approximations.

Questions on Chapter II.

1. Define "successive approximations."
2. Give the rule for extraction of square root.
3. Is any set of simultaneous equations solvable by successive approximations?
4. What is the test of solvability?
5. What are the two chief advantages of this method?

[1] When it is required to interpolate forward *from the first item* in a table, e. g., *downward* from the $t=60$ in the table above, the formula for "advancing differences" must be used with the corresponding values of y, a, b, and c; e. g., in this case, $y=1{,}569.4$; $a=-517.0$; $b=-22.9$; and $c=+3.7$. The "advancing difference" formula may be obtained from 6 by changing the sign of every Δt throughout the equation, and of the odd-ordered differences, a and c. See Supplement H.

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CHAPTER III.

EFFECT OF DIFFERENTIAL VARIATIONS.

This chapter deals with the mathematical determination of the effect of a disturbance, on the subsequent motion of a particle which moves (except for the disturbance) according to some definite differential equations of motion.

Let us consider a particle moving in time, along a plane trajectory. "Trajectory" is here used in the general sense of the path of any moving particle, rather than in the specialized ballistics sense of the path of a projectile, or the still further specialized sense which will be the meaning employed in later chapters, namely, the path of a projectile moving under certain so-called standard conditions as to atmosphere, wind, gravity, etc.

Reverting, then, to the motion of a particle along its trajectory, it is evident that at any given instant of time (t) there will be a uniquely corresponding value of x, y, x', y', x'', y'', etc., where x and y are the coordinates of the particle, x' and y' the two components of velocity, x'' and y'' the two components of acceleration, etc. Primes are thus seen to represent time derivatives, and will be used in that sense throughout this book. In this chapter the general symbol u may be used in place of x, y, x', y', x'', and y'', in theorems true as to any of them. Hence u, also, is a function of t.

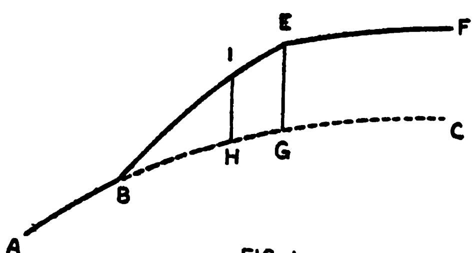

FIG. 1

In the illustrative examples of this chapter, t,x coordinates and t,y coordinates will frequently be employed, in order that the student may become accustomed to considering the elements of the trajectory as separately plotted against time, and to using time derivatives.

In the case of motion of the sort which will be considered in this book, and in fact in the case of most motions, the values of x'', y'', and higher derivatives are, in the absence of disturbing causes, determined, for any instant t, by the values of x, y, x' and y', or some of them; and hence x'', y'', etc., need not be discussed for the present.

The path of the particle will, of course, be a single curve, which may be graphed by plotting y against x. But a chronological record of its

motion may be represented more completely by *four* curves, obtained by plotting x, y, x', and y', respectively, against t.

Consider now any one of these four curves, represented by ABC in figure 1.

Suppose a disturbance from B to E, so that the curve takes the shape BE during the interval of disturbance, and suppose that thereafter there is no further disturbance and the form of the curve is EF. This is the general case. The amount of the total disturbance up to any instant is the difference in ordinates between the original undisturbed curve ABC and the disturbed curve BEF, such as HI or GE.

For convenience, disturbances may be treated as of three sorts:

(*a*) Those disturbances which produce a finite effect in a single instant; as, for instance, if the curve took the shape $ABGEF$.

(*b*) Those disturbances which vary during the total time of disturbance, so that if the total time be divided up into an infinite number of equal infinitesimal time intervals (dt_Δ), an infinitesimal part ($d\delta u$) of the total disturbance (δu) will occur during each dt_Δ, and the total disturbance may be represented as:

$$\delta u = \int_{t_0}^{T} d\delta u, \text{ or as}$$

$$\delta u = \int_{t_0}^{T} \frac{d\delta u}{dt_\Delta} dt_\Delta;$$

where t_0 is the time the disturbance starts and T is the time it ends. This is the most general case.

(*c*) Those disturbances of which a proportional part occurs during any part of the total time. These may be regarded as a special case of b.

$$d\delta u = c_1\, dt_\Delta$$

$$\delta u = c_1\,(T - t_0),$$

where c_1 is some constant.[1]

It is essential that one element of the trajectory be considered as remaining unvaried, so as to furnish a basis for measuring the variations of the other elements. Accordingly time will be selected for this purpose. Both the time (t) of which the other elements (x, y, x', y', x'', y'', etc.) are functions, and the time (t_Δ) at which a disturbance occurs, will be considered as unaffected by the disturbance. Accordingly we can say that:

$$\delta t = 0$$

$$\delta t_\Delta = 0$$

[1] As here given, c is a special case of b. But it is possible to regard both b and a as special cases of c. Thus, if the constant c_1 becomes the variable $\frac{d\delta u}{dt_\Delta}$, we have case b; whereas, if the time interval $T - t_0$ becomes infinitesimal and δu remains finite, we have case a.

and, differentiating:

7.
$$\begin{cases} \dfrac{d\delta t}{dt} = 0 \\ \dfrac{d\delta t_\Delta}{dt_\Delta} = 0 \end{cases}$$

Let us first consider disturbances of the first sort. A particle is moving through space according to some definite law of motion, expressed by differential equations. At a given instant of time ($t_\Delta = t_0$) an instantaneous disturbance takes place, which changes the value of x, y, x', and y', or some of them. Thereafter, the particle proceeds according to its original law of motion, of course not on a continuation of its original regular curve, but on another such regular curve, also satisfying the original differential equations.

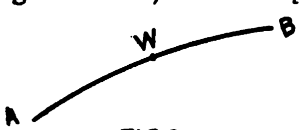

FIG.2

The amount of the instantaneous changes in x, y, x', and y' will be designated, respectively, by δx, δy, $\delta x'$, and $\delta y'$. Any one of these expressions can represent either a positive or a negative change. The changes which will be considered in any practical application of the principles of this chapter, and the effects resulting therefrom, are so minute in comparison with the elements affected by these changes, that the following restrictive definition can be given: δx, δy, etc., are small finite increments of x, y, etc., and are so minute that second and higher order terms (such as $\delta\delta x$ or $\delta x \delta y$, for instance) are of no consequence, as compared with x, y, etc., and hence any terms containing them may be disregarded and dropped from a sum or series in which first order terms occur.

The operator δ is the operator employed in that branch of mathematics known as the calculus of variations, but not always with the above restriction. No further understanding of the principles and methods of this branch than here given is necessary to the purposes of this book.

A certain resemblance between the operator δ and the familiar differential operator d will be noted. The distinction between the two should also be noted. Consider a curve in an x,y coordinate system. At any point on this curve, dy represents a continuous infinitesimal change in y along the curve, corresponding to an infinitesimal change in x. $\dfrac{dy}{dx}$ is the slope of the curve at the point in question. δy and δx represent a very small finite break in the curve in question. dy and dx may be considered as taking place in an infinitesimal period of time (dt); whereas δy and δx may be considered as taking place instantaneously, or as cumulating during a finite time interval.

Let us now derive some of the basio theorems relative to the operator δ.

8. *Independent changes in u and u' may be made at any time t_Δ.*

Proof: Consider the curve AB in x,t coordinates, and W any given point thereon. (See Fig. 2.)

The curve can be moved up or down, thus changing the x of W, without changing its t or its slope. Or the curve can be rotated about W, thus changing its slope, without changing its t or its x. Thus δx and $\delta\frac{dx}{dt}$ are independent. *Q. E. D.*

Similarly, by plotting y against t, and y against x, it can be shown that δx, $\delta x'$, δy, and $\delta y'$ are all independent.

9. *Terms containing more than one δ may be dropped.*

This was one of the fundamental hypotheses of the definition of the operator δ on page 20.

It will now be demonstrated that the four general formulas of differentiation still hold true when the operator δ replaces the operator d; in other words, that the small finite increments of this chapter obey certain laws, already familiar in form for the case of differentials, although, of course, these increments are quite different from differentials.

These four theorems are as follows:

10. $$\delta\,(cu) = c\,\delta u.$$

11. $$\delta\,(u+v) = \delta u + \delta v.$$

12. $$\delta\,(uv) = v\,\delta u + u\,\delta v.$$

13. $$\delta\left(\frac{u}{v}\right) = \frac{v\,\delta u - u\,\delta v}{v^2}.$$

Only one of these (namely, 13) will here be proved, the proof of the other three being reserved for problems.

Proof of theorem 13:

Take the expression $\frac{u}{v}$, and give u and v each an increment. Then:

$$\delta\left(\frac{u}{v}\right) = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} = \frac{v\,\delta u - u\,\delta v}{v^2 + v\,\delta v}.$$

Expand this fraction by dividing the numerator by the denominator, as follows:

$$\frac{v\,\delta u - u\,\delta v}{v^2 + v\,\delta v} = \frac{v\,\delta u - u\,\delta v}{v^2} - \frac{\delta v\,\delta u}{v^2} + \frac{u\,\delta v\,\delta v}{v^3} + \ldots$$

From the right member all terms, except the first, may be dropped, by theorem 9. Therefore:

$$\delta\left(\frac{u}{v}\right)=\frac{v\ \delta u-u\ \delta v}{v^2} \qquad Q.\ E.\ D.$$

Next let us derive the expression for "total increment," analagous to the expression for "total differential" (formula 2 of Chap. I).

14. *If u is a function X and Y, then:*

$$\delta u=\frac{\partial u}{\partial X}\ \delta X+\frac{\partial u}{\partial Y}\ \delta Y.$$

Proof: Let $u=f\,(X,\ Y)$. Give X and Y the increments δX and δY, respectively. Then:

$$\delta u=f\,(X+\delta X,\ Y+\delta Y)-f\,(X,\ Y).$$

Subtract and add the quantity $f\,(X,\ Y+\delta Y)$. Then:

$$\delta u=f\,(X+\delta X,\ Y+\delta Y)-f\,(X,\ Y+\delta Y)+f\,(X,\ Y+\delta Y)-f\,(X,\ Y).$$

Applying the law of the mean (see Osgood, p. 230) to each of these two differences gives:

$$\delta u=\frac{\partial f(x+\theta_1\,\delta X,\ Y+\delta Y)}{\partial X}\delta X+\frac{\partial f(X,\ Y+\theta_2\,\delta Y)}{\partial Y}\delta Y.$$

Now, if these two partial derivatives are continuous, each would approach the corresponding partial derivative of f (X, Y) if δX and δY both were to approach zero, and hence will differ but slightly when δx and δy are very minute.

Consequently we may express

$$\frac{\partial f(X+\theta_1\delta X,\ Y+\delta Y)}{\partial X}\ \text{as}\ \frac{\partial f(X,\ Y)}{\partial X}+\delta\ \frac{\partial f(X,\ Y)}{\partial X}.$$

and

$$\frac{\partial f(X,\ Y+\theta_2\,\delta Y)}{\partial Y}\ \text{as}\ \frac{\partial f(X,\ Y)}{\partial X}+\delta\ \frac{\partial f(X,\ Y)}{\partial Y}.$$

Substituting these values in the expression for δu, and substituting u for $f\,(X,\ Y)$, gives:

$$\delta u=\frac{\partial u}{\partial X}\ \delta X+\frac{\partial u}{\partial Y}\ \delta Y+\delta X\ \delta\frac{\partial u}{\partial X}+\delta Y\ \delta\frac{\partial u}{\partial Y}$$

from which the last two terms can be dropped by theorem 9.

An extension of this derivation gives:

$$\delta u=\frac{\partial u}{\partial X}\ \delta X+\frac{\partial u}{\partial Y}\ \delta Y+\frac{\partial u}{\partial Z}\ \delta Z+\ \cdot\ \cdot\ \cdot \qquad Q.\ E.\ D.$$

15. $$\frac{d}{dt_\Delta}(\delta u)=\dot{\delta}(u')$$

Proof: Consider a particle moving along a curve from A to P. When it reaches P, at time t_1, let x and y instantaneously receive an increment (δx and δy, respectively) which will place the particle at p, and thereafter let the particle move undisturbed along the curve pD (the curves AP and pD being defined by the same differential equations of motion). Let Q be a point on the curve AP, such that, if no disturbance had taken place at P, the particle would have reached Q after a small finite time interval Δt from the time it left P. Let q be the point on the curve pD reached by the particle, Δt seconds after leaving p. Then a pair of changes $\delta x + \Delta\delta x$ and $\delta y + \Delta\delta y$, occurring at time $t_1 + \Delta t$, would produce the same effect as the pair of changes δx and δy occurring at time t_1.

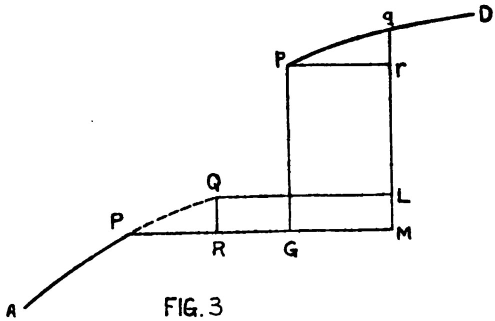

FIG. 3

Let δx and δy, although arbitrary at time t_1, be thereafter considered as restricted by the condition that at any instant thereafter their value must be such as to produce at that instant the same situation as would have existed at that instant, had the changes been made, with their initial values, at time t_1. From this point of view, the average rate of change of δx and δy, during the interval Δt, is respectively $\frac{\Delta\delta x}{\Delta t}$ and $\frac{\Delta\delta y}{\Delta t}$. Δt is here regarded as a change in the time of disturbance (t_Δ).

$$\Delta\delta x = RM - PG = GM - PR = pr - PR;$$

$$\Delta\delta y = Lq - Mr = rq - ML = rq - RQ.$$

Let Δt approach zero. The rates of change at time t_1 thus become $\frac{d\delta x}{dt_\Delta}$ and $\frac{d\delta y}{dt_\Delta}$.

$$\frac{d\delta x}{dt_\Delta} = \lim \frac{\Delta\delta x}{\Delta t} = \lim \frac{pr - PR}{\Delta t}$$

$$\frac{d\delta y}{dt_\Delta} = \lim \frac{\Delta\delta y}{\Delta t} = \lim \frac{rq - RQ}{\Delta t}$$

Now, the unaffected x-component of velocity (x') at time t_1 is the limit of $\frac{PR}{\Delta t}$, as Δt approaches zero, Δt being here regarded simply as a change in the time (t) of which the elements of the trajectory are

functions. The value of x' at time t_1 as affected by the variations, is $\frac{pr}{\Delta t}$. Therefore:

$$\delta(x') = \lim\left(\frac{pr}{\Delta t} - \frac{PR}{\Delta t}\right) = \frac{d\delta x}{dt_\Delta}$$

Similarly:

$$\delta(y') = \lim\left(\frac{rq}{\Delta t} - \frac{RQ}{\Delta t}\right) = \frac{d\delta y}{dt_\Delta} \qquad Q. E. D.$$

16. *The operator δ does not always signify an* **independent** *increment.*

Examples: In the expression for the total increment given under theorem 14, if any three of the four increments are conceived of as independent of each other, then the fourth must of necessity be dependent upon the other three.

In equation 38 in Chapter IX, δX may be taken as a constant, and δx as dependent on δy, $\delta x'$, and $\delta y'$.

Throughout this book, δt and δt_Δ will be taken as zero, by theorem 7.

So we may say that changes represented with the operator δ are essentially independent, unless rendered otherwise by some express condition of the problem confronting us.

Let us now consider the following problem. If a particle, moving according to some definite law up to time t_Δ, suffers a disturbance and then moves on according to the original law, without further disturbance, what is the relation between the value of its x at some later time T and the value which x would have had at time T if there had been no disturbance? Let X represent the undisturbed value of x at time T, and let $X+\delta X$ represent the disturbed value. The problem is to express δX in terms of the δx, δy, $\delta x'$, and $\delta y'$ occurring at time t_Δ.

If y, x', and y' are not changed at time t_Δ, then δX is of the form $L\,\delta x$, plus terms containing more than one δ, which can accordingly be dropped by theorem 9. L will have a value which will depend on the trajectory in question and on the point on that trajectory at which the change δx occurred. Thus for any given trajectory L is a function of t_Δ, but is not dependent on δx.

Similarly, if x, x', and y' are not changed at time t_Δ, the resulting δX can be expressed as $M\delta y$, etc.

If all four of x, y, x', and y' are changed at time t_Δ, the resulting δX will equal $L\,\delta x + M\,\delta y + N\,\delta x' + P\,\delta y'$, plus terms containing combinations of δx, δy, etc., which may therefore be dropped by theorem 9.

Thus we have, as the fundamental equation for the X-effect at time T, due to a set of small arbitrary changes in x, y, x', and y' at time t_Δ:

17. $$\delta X = L\,\delta x + M\,\delta y + N\,\delta x' + P\,\delta y'$$

Formal expressions for L, M, N, and P may be obtained from theorem 14, but have not yet been shown to be of any practical value.

The fact the δx, δy, $\delta x'$, and $\delta y'$ can each be given any arbitrary value at any time of change (t_Δ) enables us, if we wish, to assign any arbitrary value we please to δX and any three of these, and then satisfy the equation by a proper choice of value for the fourth.

18. δX *is not a function of* t_Δ.

From the viewpoint of theorem 15, the values of δx, δy, etc., although initially arbitrary, are regarded as restricted so as to change value at such a rate, during the interval dt_Δ, as not to alter the resulting value of δX. *Q. E. D.*

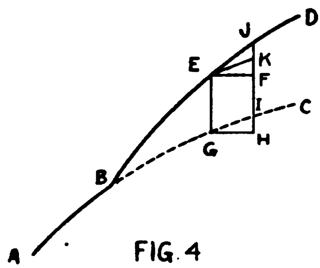

FIG. 4

Thus far, we have been considering merely instantaneous changes. Let us now consider the second sort of changes listed at the beginning of this chapter, namely changes which vary throughout a finite time, starting as zero at the beginning of the interval. A velocity change of this latter sort is readily seen to be made up of an infinite number of infinitesimal changes of velocity throughout the interval. Similarly a coordinate change of this sort is seen to be made up of an infinite number of infinitesimal changes in position throughout the interval.

Consider the finite time interval, from time t_0 to time T, as divided up into an infinite number of infinitesimal time intervals, dt_Δ. An infinitesimal part of the total δx, δy, $\delta x'$, and $\delta y'$ occurs during each of these infinitesimal intervals, and thus causes an infinitesimal part of the total δX which will occur at time T. Thus the result, at time T, of the disturbance during any interval dt_Δ, is

19. $$d\delta X = L\,d\delta x + M\,d\delta y + N\,d\delta x' + P\,d\delta y'.$$

It should be noted that $d\delta x$, etc., are here used in quite a different sense from that of theorems 15 and 16. Here $d\delta x$ means the change which δx undergoes along the actual disturbed path of the particle, during the interval dt_Δ. But here also, d and δ may be shown to be commutative, as in theorem 15.

Consider x plotted against t. Let ABC, Fig. 4, represent the t,x curve of the undisturbed trajectory. Let a disturbance start at B changing the shape of this curve to BED. Let EF ($=GH$) represent one of the infinitesimal time intervals, dt_Δ.

The normal undisturbed rate of change of x is HI. The actual disturbed rate of change of x is FJ. Thus:

$$HI = dx$$

$$FJ = dx + \delta dx$$

Let EK be parallel to GI.

The total δx at the beginning of the dt_Δ interval is GE. The total δx at the end of the interval is IJ. Thus:

$$d\delta x = IJ - GE = IJ - IK = KJ$$

But:

$$\delta dx = FJ - HI = FJ - FK = KJ$$

Therefore:

$$\frac{d\delta x}{dt_\Delta} = \delta\left(\frac{dx}{dt}\right) \qquad Q.\ E.\ D.$$

The right-hand member of formula 19 can now be transformed by transposing d and δ, and by both dividing and multiplying each term by dt_Δ, and by applying this principle of commutativeness.

20. $$d\delta X = L\,\delta x' dt_\Delta + M\,\delta y' dt_\Delta + N\,\delta x'' dt^\Delta + P\,\delta y'' dt_\Delta$$

Integrating this from t_0 to T, we get

21. $$\delta X = \int_{t_0}^{T} L\,\delta x' dt_\Delta + \int_{t_0}^{T} M\,\delta y' dt_\Delta + \int_{t_0}^{T} N\,\delta x'' dt_\Delta + \int_{t_0}^{T} P\,\delta y'' dt_\Delta$$

This is the formula for the effect, at time T, of changes which vary throughout the finite time interval from t_0 to T.

For solution of any of the integrals, the $\delta x'$, $\delta y'$, $\delta x''$ or $\delta y''$ therein contained must be replaceable by something which is constant either in value or in algebraic form throughout the interval.

Let us now consider the third sort of changes listed at the beginning of this chapter, namely, changes which are proportional to time. Consider a δx of this sort. Then, from formula 19:

$$d\delta X = L\,d\delta x.$$

Substituting dt_Δ for $d\delta x$, and integrating within the limits of the disturbance, we get:

$$\delta X = \int_{t_0}^{T} L\,dt_\Delta$$

If L is a constant, this becomes

$$\delta X = L\,(T - t_0).$$

PROBLEMS.

(19) Prove theorem 10.

(20) Prove theorem 11.

(21) Prove theorem 12.

(22) Prove that $\delta \sin u = \cos u \, \delta u$ and that $\delta \cos u = -\sin u \, \delta u$. Suggestion: Turn to some book on the calculus, and follow the analogy of the similar differential formulas.

QUESTIONS ON CHAPTER III.

1. What is the object of this chapter?
2. What is the meaning of the operator δ in ballistics?
3. Do formulas involving the operator d hold true with respect to the operator δ?
4. When would $\frac{\delta y}{\delta x}$ represent the slope of a curve?
5. In this chapter what is meant by the word "trajectory"?
6. Distinguish between the two meanings of $d\delta u$ and δdu.
7. What variations of time are considered in this chapter?
8. Is δu an infinitesimal?
9. Distinguish between δu and du.
10. Interpret M of formula 17 by means of formula 14.

CHAPTER IV.

FINITE DIFFERENCES.

Integration by finite differences is based upon the principle that inasmuch as a derivative is a rate of change, the value of an integral of any smooth function can be computed step by step, if successive values of its derivative are known at sufficiently close intervals.

Suppose that u is a function of t, such that u can not be integrated with respect to t by means of any of the expressions tabulated in any table of integrals. In other words, formal integration is impossible.

Let us now imagine a plotted curve with u for its ordinates and t for its abcissas. Then $\int^{t_m} u\,dt$ is the area between the curve, the t axis, and the ordinates at $t=t_n$ and $t=t_m$. Such as area exists for every continuous curve, and hence every continuous curve has an integral, even though that integral is not expressible by means of the ordinary elementary functions.

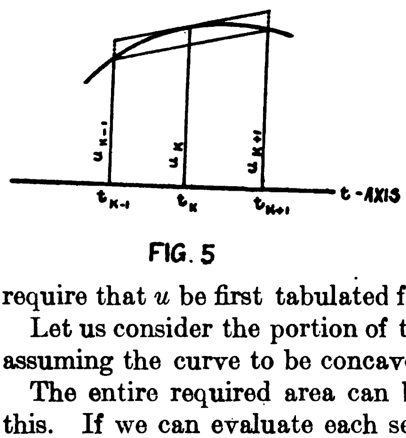

FIG. 5

In such a case, various approximations are possible, some crude and some so refined that they can produce results to any desired degree of precision.

All of the methods here described require that u be first tabulated for successive finite values of t.

Let us consider the portion of the curve lying between t_{k-1} and t_{k+1}, assuming the curve to be concave downward.

The entire required area can be divided up into sections such as this. If we can evaluate each section, the sum of these evaluations will be the value of the whole.

A first approximation would be the area under the chord. Thus:

$$A_1 = \frac{u_{k-1} + u_{k+1}}{2}\,(t_{k+1} - t_{k-1})$$

If we space t with unit intervals, then:

22. $$A_1 = u_{k-1} + u_{k+1}$$

This approximation is too small.

A second approximation would be the area under the tangent at the point (u_k, t_k). Thus:

$$A_2 = u_k\,(t_{k+1} - t_{k-1})$$

If we space t with unit intervals, then:

23. $$A_2 = 2u_k$$

This approximation is too large, but the error is about one-half the error of A_1. A, the true area, lies between A_1 and A_2. Thus:

$$A_2 > A > A_1$$

If the curve were concave upward, the above inequality would be reversed. In either case it is evident by inspection that:

$$A \approx \frac{2A_2 + A_1}{3}$$

Therefore a third, and very close, approximation is:

24. $$A_3 = \frac{2A_2 + A_1}{3} = \frac{1}{3}\,(u_{k-1} + 4u_k + u_{k+1}).$$

This is known as Simpson's rule.[1]

PROBLEMS.

Evaluate the following to four decimal places, by formal integration, and by formulas 22, 23, and 24, and determine the percentage of error in each case:

(23) $\int_8^{10} x^2\,dx$

(24) $\int_3 \frac{dx}{x}$

In using Simpson's rule, when the tabular interval is other than unity, the formula is:

25. $$A_3 = \frac{h}{3}\,(u_{k-1} + 4u_k + u_{k+1})$$

[1] Formula 24 can be proved as follows: Any ordinary function of one variable can be expressed as a power-series of that variable. Thus the general curve which we have been considering so far in this chapter can be expressed:

$$u = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4 + \dots$$

This equation represents a straight line, another straight line, a quadratic, a cubic, a quartic, etc., according as we drop all but the first, all but the first two, etc., terms of the right member of the equation. For all approximations up to and including a cubic, formula 24 is precise. Proof: By this formula,

$$\int_0^{2h} u\,dt = \frac{h}{3}(u_0 + 4u_h + u_{2h})$$

By formal integration.

$$\int_0^{2h} u\,dt = \left[p_0 t + \frac{p_1 t^2}{2} + \frac{p_2 t^3}{3} + \frac{p_3 t^4}{4}\right]_{t=2h}$$

These are identical. With h sufficiently small, similar pairs of expressions, involving quartics or higher, are nearly mutually identical, being indeed coincident in as far as the first four terms are concerned, the discrepancy in the higher terms being very minute. *Q. E. D.*

For another proof, see Osgood, "*Differential and Integral Calculus,*" *1917*, pages 406–408.

where h is the tabular interval. The area of the next section will be $\frac{h}{3}(u_{k+1}+4u_{k+2}+u_{k+3})$ and so on, so that the area for a series of sections will be $\frac{h}{3}(u_{k-1}+4u_k+2u_{k+1}+4u_{k+2}+2u_{k+3} \cdot \cdot \cdot 4u_{n-1}+u_n)$.

PROBLEMS.

(25) Divide $\int_0^1 z\,dx$ up into ten sections, where $z=x-x_2$. Tabulate z against x, and integrate by Simpson's rule. Integrate formally, and compare the results.

(26) u is a function of t:

$$u=10\left(2+\frac{t}{10}\right)^4$$

Tabulate the values of u corresponding to $t=0, 1, 2 \ldots 9, 10$. Evaluate $\int_0^{10} u\,dt$ by Simpson's rule and by formal integration, and compare the results.

(27) Evaluate $\int_0^{10}\left[1000 \log_{10}\left(10+\frac{t}{10}\right)-1000\right]dt$, either by formal integration, or by tabulating from a denary log table and then using Simpson's rule.

Simpson's rule is a method of *numerical* integration, as distinguished from *formal* integration. It is one of the simplest of a system of rules that may be obtained, involving the values of the function and its differences. Let us now derive a method of numerical integration which employs finite differences of various orders.

We will suppose a tabular function of t and will tabulate its first second, third, and fourth receding differences as follows, when each difference as tabulated is obtained by subtracting the element on the line above from the element on the same line of the preceding column, i. e., where $a_0=f_0-f_{-1}$; $b_0=a_0-a_{-1}$; etc.[2]

t	$f(t)$	First difference.	Second difference.	Third difference.	Fourth difference.
-4	f_{-4}				
-3	f_{-3}	a_{-3}			
-2	f_{-2}	a_{-2}	b_{-2}		
-1	f_{-1}	a_{-1}	b_{-1}	c_{-1}	
0	f_0	a_0	b_0	c_0	d_0

[2] This rule should be followed regardless of the order in which the function is tabulated, i. e., regardless of whether t increases or decreases down (or from left to right across) the page. Thus, if the tabular function of t algebraically increases as one goes down (or from left to right across) the page, the first difference will be *positive;* if *decreasing*, it will be *negative*. Similarly, if the first difference algebraically *increases*, the second difference will be *positive;* if *decreasing*, *negative*, etc.

These differences are called "receding," for the reason that the differences tabulated on any line would have receded if they had been tabulated opposite the space between the two elements of which they are the difference; sometimes, for clearness, they are so tabulated; but it is generally more convenient to tabulate them as above.

(28) Successively evaluate f_{-1}, f_{-2}, f_{-3}, and f_{-4} in terms of f_0, a_0, b_0, c_0, and d_0.

(29) From this deduce a formula for f_{-n}.

(30) Substitute t for $-n$. The result is the usual interpolation formula in terms of receding differences, whereby $f(t)$ can be calculated with values for any fractional value of t. The resulting $f(t)$ will lie on a smooth curve within the interpolation interval and, if the true $f(t)$ be smooth, will closely approximate it.

(31) What is the interpolation formula for $t=\frac{1}{2}$; for $t=-\frac{1}{2}$?

(32) In the tabulation of problem 27, what is the logarithm of 10.55? Use either interpolation formula from the preceding problem.

(33) Integrate the formula of problem 30, between the limits -1 and 0.

(34) Integrate it between 0 and 1.

(35) Integrate it between -1 and 1.

The solution of the last three problems gives us, respectively, the formula for integrating by finite differences and two formulas for integrating ahead. These are the fundamental formulas of the method of computing trajectories by numerical integration, which method will be discussed in Chapter VIII.

(36) Form the differences of the tabulation in problem 26 and integrate each interval "ahead," using the formula of problem 34 in the earlier stages, and that of problem 35 in the later stages.

(37) Integrate this same tabulation "across," by using the formula of problem 33. Compare the two sets of results.[3]

(38) Sum up the results in problem 37 and compare with the two answers already obtained for problem 26.

---

[3] In numerical integration the values of the first few integrals will be inexact, due to the absence of first, second, third, etc., differences in the integrand. To supply this lack extrapolate back for the missing differences from such first, second, third, etc., differences as are obtainable from the tabulation. A better method would be to rewrite the first four or five of the tabulated functions in reverse order, and difference this new series as usual. What was the first interval being now the last, the set of differences to be used in integrating over this last interval by the formula of problem 33 are now obtainable. The figures now on the last line will be found to be the figures that were at the top of the columns of the original tabulation, except that the signs of the differences of odd order are now changed. By the method here suggested the student can construct for himself formulas for interpolating and integrating at the beginning of a table with these *advancing* differences. When this is done, the rewriting of the function in reverse order may be discontinued. One of the methods of procedure outlined in this note should be followed in this and in all succeeding problems of numerical integration. See Supplement H.

(39) Substitute 1 for t in the interpolation formula of problem 30. The result is an extrapolation formula.

(40) Using the values $f_{-1}=f_0-a_0$, $f_0=f_0$, and the value of f_1 obtained by the preceding problem; evaluate $\int_{-1}^{0} f\,dt$, $\int_{0}^{1} f\,dt$, and $\int_{-1}^{1} f\,dt$ by Simpson's rule, obtaining the values of $f_{-\frac{1}{2}}$ and $f_{\frac{1}{2}}$ by the formulas of problem 31. Compare the resulting formulas with those obtained by problems 33, 34, and 35.

(41) Derive formula 6 used in problem 18 of Chapter II.

(42) The last tabular values of t, y, a, b, and c for a time interval of one second are

t	y	a	b	c
41	−23.0	−2501	−8.2	+25

Find the value of t corresponding to $y=0$, using the formula of problem 41. Interpolate for y as a check, using the formula of problem 30. Carry t to four decimal places.

(43) The following tabulated values of y'' and x'' from an actual trajectory computation are given: 8-inch rifle, railway mount, model of 1917. Muzzle velocity (V), 594.36 m/s; ballistic coefficient (C), 4.0; angle of departure (ϕ), 10°. The meaning of these terms need not be considered at this stage.

Values of x'' and y'' at two-second intervals:

x''	t	y''
−36.8	0	−16.3
−29.4	2	−13.9
−23.6	4	−12.2
−18.9	6	−10.9
−15.1	8	− 9.9
−11.9	10	− 9.3
− 9.4	12	− 8.9
− 7.5	14	− 8.6
− 6.3	16	− 8.4
− 5.4	18	− 8.2

Obtain y', x', y, and x for each value of t by numerical integration, checking the final results by Simpson's rule. Initially $x=0$, $x'=V\cos\phi$; $y=0$, $y'=V\sin\phi$.

(44) From this data, calculate values for t, x, x', and y' corresponding to $y=0$. As a check, get the value of y corresponding to this value of t.

The formulas derived in problems 28 to 35 inclusive, and in problem 38, are to be used *only* with *receding* differences, and should be plainly so marked in the student's notebook.[4]

Questions on Chapter IV.

1. Define "integration by finite differences."
2. State "Simpson's rule."
3. How are first, second, etc., differences formed in the case of a tabular function?
4. What is the distinction between "numerical integration" and "formal integration"?
5. From theoretical considerations rather than from the numerical results which happen to have been obtained in any of the numerical problems, what is your opinion of the relative precision of the six methods of numerical integration given in this chapter? State detailed reasoning.
6. State the three formulas for integration by receding differences.
7. When should each be used?
8. What is meant by receding differences?
9. Give two methods of integrating at the beginning of a table.

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[4] For methods of deriving advancing difference formulas, see Supplement H.

CHAPTER V.

ELEMENTS OF THE TRAJECTORY.

From the point of view of the present state of ballistics, the following are the more important elements and features of a trajectory:

Trajectory.—This term, as here used in connection with computations, embraces only the standard trajectory of the projectile, moving in accordance with assumptions which may be laid down as follows:

(1) The earth is motionless. (The average effect of the earth's rotation on gravity is included in the assumed value of g.)

(2) The gun and target are in the same altitude above sea level.

(3) The preassigned standard muzzle velocity for that type of gun is actually obtained.

(4) There is no wind.

(5) The atmospheric density varies regularly with the altitude according to the exponential law assumed and is standard at the muzzle (15° C. = 59° F. 750 mm. of mercury, 78 per cent saturation; 1.2034 kg/m^3).

(6) The action of gravity is uniform in intensity, is directed towards the earth's center, and is independent of the geographical location of the gun, $g = 9.80$ m/s^2.

(7) In the computation of standard trajectories, the velocity-resistance factor, $G(v)$, may be regarded as dependent only on the velocity; the standard variation in elasticity of the air being accounted for by the value of the ballistic coefficient adopted.

(8) The ballistic coefficient is a constant on the trajectory and is as determined from experimental firing. This includes the assumption that all precessional and nutational effects of the projectile may be ignored in computation, and hence the trajectory as computed lies in a vertical plane.

(9) The vertical jump of the gun is as assumed from observation.

(10) The atmospheric temperature (and hence elasticity) varies regularly with the altitude according to the law assumed, and is standard at the muzzle (15° C. = 59° F.).[1]

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[1] A question arises in connection with assumptions (7), (8), and (10) as to the use of the temperature structure. The standard temperature structure adopted by the Ordnance Department is supposed to approximate average conditions aloft. Firing conditions on different occasions can be accurately compared only when reduced to the same standard temperature structure, and for convenience the above-mentioned standard temperature structure is always used as the basis of comparison. Variations in range due to irregularities in temperature aloft are computed by reference to this standard temperature structure. But the G function used in the computation of trajectories for standard conditions is assumed to be dependent only on the velocity. This amounts to assuming, for the trajectory computation, that the temperature is the same for all altitudes. The G function was based on a number of actual firings which were not corrected for variations in temperature. These firings were probably made under average surface conditions, that is, for temperature approximately 15° C.=59° F. The computed trajectories involve a ballistic coefficient, C, which is so taken as to make the range check with firings corrected to standard temperature structure aloft. The explicit introduction of variable temperature in the original computations, while possible for individual trajectory computations, is not regarded as important, and is not feasible in the case of the ballistic tables.

(11) The drift (including lateral jump) is as assumed from observation.

Coordinates (*x and y*).—The coordinates of any point on the trajectory, measured in meters. The abcissas (x) are measured along the surface of the earth and are positive in the direction of fire. The ordinates (y) are measured vertically from the surface of the earth and are positive upward. The origin is the muzzle of the gun. In the development of formulas and in computation, this coordinate system is treated as cartesian, the error being negligible. Note the difference between this conception and the tangent-plane conception of the Ingalls-Siacci ballistics. The present conception, being based on a curved earth, obviates the necessity for correcting for curvature of the earth.

Surface of earth.—A spherical surface passing through the muzzle of the gun and concentric with the earth.

Muzzle velocity (V).—A fictitious initial tangential velocity of the projectile at the beginning of its flight in meters per second. The blast may continue to accelerate the projectile for some distance beyond the muzzle; so that the "muzzle velocity" is not the actual velocity at the muzzle, but is rather a fictitious velocity which, if it occurred at the muzzle and if there were no blast, would cause the projectile to travel on the same trajectory as that on which it actually travels.

Velocity (v).—Sometimes called "remaining velocity." The tangential velocity of the projectile at any point of its flight, in meters per second. The x and y components of the velocity are represented by x' and y' respectively.

Acceleration (x'' *or* y'').—The rate of increase of x' and y', respectively (in meters per second per second), at any point on the trajectory.

Time (t).—The time in seconds elapsed in the flight of the projectile from the muzzle to any point on the trajectory. t is the independent variable of the trajectory. (When considered as the time when a disturbance takes place, the symbol is t_Δ.)

Quadrant angle of departure (ϕ).—The angle measured from the horizontal to the tangent to the trajectory at the muzzle; sometimes called the angle of projection.

Inclination (θ).—The angle measured from the horizontal to the tangent to the trajectory at any point on the trajectory.

Point of fall (*under range table assumptions*).—The point where the projectile in its downward flight reaches the same altitude above sea level as the muzzle.

(Geographical) range (X).—The distance in meters from the muzzle to the point of fall, measured on the surface of the earth.

Quadrant angle of fall (ω).—The negative of the inclination at the point of fall.

Retardation (R).—The retardation due to the resistance of air of standard density and elasticity. R is a function of velocity and altitude.

$$R = vE$$

$$E = \frac{GH}{C}$$

E is called the resistance function; G is a tabular function of v, for convenience tabulated to argument $\frac{v^2}{100}$; C is the ballistic coefficient; H is determined by the following exponential law:

$$H = e^{-hy}$$

$$h = .0001036$$

Quadrant elevation.—The angle between the horizontal and the axis of the bore just before the gun is fired.

Vertical jump.—The algebraic difference obtained by subtracting the quadrant elevation from the quadrant angle of departure.

Summit.—The highest point of the trajectory.

Maximum ordinate (y_s).—The y coordinate of the summit.

Time of flight (T).—The time (t) from the muzzle to the point of fall.

Angle of site.—The angle whose tangent is the ratio of the difference in altitude of gun and target divided by the range. The angle is positive if the gun is higher than the target. Owing to the fact that, in the present ballistics usage, zero altitude is considered as being the same elevation above sea level as the gun, rather than as lying in the plane horizontal at the gun, the angle measured *directly* with a transit or level is only *approximately* equal to the angle of site, although satisfactory for short ranges.

Point of splash.—The point where the projectile, in range firing, enters the water.

Center of impact.—The average position of several points of splash.

Ballistic coefficient (C).—A purely empirical number, used in the formula for the resistance function (E). C is a mean value (constant over a given trajectory) of the reciprocal of the relative retardation. The relative retardation is the ratio of the retardation experienced by the actual projectile to that which would have been experienced by a certain fictitious "standard projectile," moving at the same velocity and altitude. C for any given projectile and muzzle velocity varies only as a function of the angle of departure. C may be represented as:

$$C = \frac{w}{id^2}$$

where w is the weight of the projectile in pounds, d its diameter in inches, and i the "coefficient of form," so-called because it is largely

dependent upon the form of the shell. This i is an empirical number, its value being determined by giving the C in the equation above the value necessary to make the range computed with that C equal to the actually observed range, when the latter is reduced to standard conditions.

Ascending branch.—That part of the trajectory in which the projectile rises.

Descending branch.—That part of the trajectory in which the projectile descends.

Plane of projection.—The vertical plane including the line of projection, i. e., the tangent to the trajectory at the muzzle.

PROBLEM.

(45) Prove that—

$$x' = v \cos \theta$$

$$y' = v \sin \theta$$

QUESTIONS ON CHAPTER V.

1. State the eleven standard trajectory assumptions.
2. Which of these assumptions are always in error?
3. What corrections in practice have to be applied to compensate for variations rom each of the ten assumptions?
4. Define E, and each of its elements, and explain how each is obtained.
5. Draw a trajectory lying in the plane of fire and label the following: Gun, surface of earth, angle of departure, elevation, angle of jump, maximum ordinate, range, point of fall, angle of fall, ascending branch, descending branch, summit. Such as have symbols may be labeled by the appropriate symbol.
6. Draw another trajectory. Mark a dot on it to represent the projectile in flight at the end of t seconds. At that point, draw an arrow to represent the velocity. Draw arrows to represent its components, x' and y'. Label x, y, and θ.
7. Define each of the elements of the two preceding questions.
8. Why does the assumption of a curved earth (see trajectory assumption No. 1, "Coordinates" and "Surface of the earth") make it impossible to measure the angle of site precisely with a transit or level?
9. Why is the assumption of a curved earth (see "Coordinates") more convenient than the assumption of a flat earth?

CHAPTER VI.

HISTORY OF EXTERIOR BALLISTICS.

The science of ballistics consists of three parts, namely: Interior ballistics, dealing with the behavior of a projectile in the gun; exterior ballistics, dealing with its behavior during flight; and ballistics of penetration, dealing with its behavior while entering the target. This book deals only with exterior ballistics.

The fundamental differential equations of motion of a projectile in flight have been known since the time of Newton. These are based on the two components of acceleration, which may be represented as follows:

$$\text{Horizontal acceleration} = -R \cos \theta$$
$$\text{Vertical acceleration} = -R \sin \theta - g$$

in which R is the retardation due to the resistance of the atmosphere, θ the inclination (to the horizontal) of a tangent to the trajectory, and g the acceleration of gravity.

These equations look very simple, and would be if R were a constant or were one of certain simple expressions, in terms of time, velocity, x, y, or θ. But R appears to follow no mechanical law which can be algebraically expressed. All that we know about R has been derived empirically, and even the simplest available approximate expression for R renders formal integration out of the question.

Two great obstacles to the development of exterior ballistics have always been (1) lack of knowledge about R and (2) the difficulty of solving the differential equations. On the basis of the methods pursued in attempting to overcome these obstacles, the progress of ballistics may be divided into three periods.

The first period, namely all the years prior to about 1865, may be called "the algebraic period." During this period, attempts were made to represent R by some simple algebraic expression, and thus to solve the equations by artifices such as those found in the standard methods of calculus. At first it was generally assumed that the motion of a projectile through the air satisfied the conditions laid down by Newton, under which R would vary as v^2. The experiments of Robins and Hutton, and the two sets of firings at Metz, furnished the first real data on R at velocities sufficient for practical ballistics. As a result, R was variously supposed to vary as the square or as the cube of v, or as some combination of the two; and, based on these

suppositions, there were devised many extremely ingenious solutions of the equations, all of which solutions were found in course of time to be insufficiently approximate, and most of which were very laborious.

The second period, lasting from about 1865 to the beginning of the present war, may be called "the period of approximations." Practically all of the physics and mathematics of this period were based upon Mayevski's formula:

$$R = \frac{Av^n}{C}$$

in which R is the retardation, v the velocity, and C the ballistic coefficient. It was assumed that velocities could be divided into a few intervals (such as 0 to 790 f/s., 790 to 990 f/s., etc.), each interval with its own law of resistance, and each with its own constant value for A and for n. The experiments of this period, notably those of Bashforth and Mayevski, and the Meppen and Gâvre firings, were accordingly directed to finding how these intervals might be taken, and what the proper constants would be for each interval. The solutions of the differential equations of motion (notably those of Bashforth, Mayevski, Zaboudski, Hojel, Siacci, Didion, Braccialini, and Ingalls), attempted during this period, not only made use of the Mayevski formula, but also, in developing the various equations, used frequent approximations, most of them based upon the assumption of a practically flat trajectory. Thus the equations were finally wrenched into a solvable form. Our familiar Ingalls tables were produced in this manner.

The third period, extending since the beginning of the World War, may be called "the period of numerical integration." Since the time of Euler it had been known that, by the use of numerical integration, the differential equations could be integrated in their *original* form without resort to approximations. But the approximate methods were always thought to be simpler and sufficiently precise, until certain phases of long-range and high-angle fire introduced during the World War compelled a resort to numerical integration. Thus ballistics has reverted to the first principles.

The chief differences between the Ingalls-Siacci ballistics and present ballistic methods may be summarized as follows:

The former used a datum plane tangent to the surface of the earth at the gun. The latter uses the actual surface of the earth as datum.

The former assumed air of a uniform density, this density being chosen at such an average for each arc as to give the correct range difference for each arc, and hence the correct range for the point of fall, but not giving the other terminal elements precisely, and producing undependable results at other points on the trajectory, especially in high-angle or long-range fire. The latter assumes the air

of a standard structure approximating closely to observed data: less dense the further one gets above the surface of the earth, in accordance with an explicit mathematical law.[1]

The former represented the trajectory by equations which had been simplified by a step which assumed (for the purposes of this step) that the trajectory is a continuous straight line. This assumption produced grave errors in high-angle fire. The latter solves the equations in their strict form as originally deduced from the laws of physics.

The former approximated the observed atmospheric resistance function by certain rather roughly continuous formulas. The latter uses the resistance as a tabulated function, as derived from experiment, and smooth throughout.[2]

The former found it very difficult to obtain explicit data without assuming[3] the rigidity of the trajectory, which of course is inadmissible if the target is considerably above or below the level of the gun. The latter is adapted to furnishing just as exact information concerning any point on the trajectory as concerning the point of fall, and hence need not assume the theory of rigidity.

The former possessed no practical method of correcting the range for variable wind aloft, or computing the weighting factors therefor, and hence often gave results so incorrect as even in exceptional cases to have the wrong sign. The latter can compute, with any desired degree of precision, the effect of any atmospheric change occurring at any point on the trajectory, on the basis of the usual physical assumptions.

In addition, modern ballistics frankly treats the ballistic coefficient as being purely empirical; and also introduces two new corrections, which become important in long-range fire, namely corrections for rotation of the earth and for changes in the elasticity of the air. The *derivation* of the present day formulas is, on the whole, as simple as the derivation of the formulas of the Ingalls-Siacci system, and their *use* in the construction of ballistic tables and range tables requires only the most elementary mathematics.

In view of the foregoing comparison, it is obvious why the later method is better adapted than the earlier to the high-angle fire, the long-range fire, and the fire at a target considerably above or below the gun (including antiaircraft fire), which predominate in modern warfare.

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[1] Cf. "Physical Bases" (Ordnance Textbook 972), p. 5.

[2] Cf. "Physical Bases" (Ordnance Textbook 972), p. 4.

[3] This assumption is: To strike a target at a different level from that of the gun, a trajectory may, *without changing its form*, be rotated vertically, about an axis through the gun. See page 68.

The numerical integration which characterizes modern ballistics is often called the "short-arc method." Short arc methods have existed in the past. That of Siacci (discarded by him) consisted in carrying his approximations only during a change of say 5° in θ, then starting a new set of approximations based on the value of θ at that point, and carrying this new set for the next change of 5° in θ, etc. But the words "short arcs" and "successive approximations" should not lead the student to assume that Siacci's method of making a series of approximations over successive short arcs bore any close resemblance to modern methods of numerical integration.

Numerical integration has long been used to compute the orbits of heavenly bodies, but was applied to the computation of trajectories in this country for the first time in 1917. The necessity for these more exact methods was realized in this country as the result of reports from England and France in which analogous developments had arisen.

The development of the present ballistic methods in this country have been covered in the introduction.

Let it not be thought, however, that present methods have relegated the familiar Ingalls tables to the discard. The Ingalls-Siacci methods were devised for guns which were rarely if ever fired at an elevation of over 15°; therefore their failure to apply to higher elevations is not to their discredit. For fire up to 8°, the French still use the old methods, even as a basis for their new tables. Until the new American ballistic tables are completed, there is no reason why the Ingalls tables should not be used for all low-angle short-range computations.

And even in the computation of higher-angle, longer-range trajectories by present methods, the Ingalls tables have an important place, for by their means the observed-range of range firings can be corrected to the range-under-standard-conditions, and the first approximate value for C can be obtained from this standard range, the muzzle velocity and the angle of departure.

Then, too, the muzzle velocity is obtained from the observed instrumental velocity by means of Ingalls' tables, which have many other practical uses in ballistic experimentation.

The object of the new methods is thus seen to be not to supplant the Ingalls tables, as being something inaccurate and obsolete; but rather to treat these tables as sufficient in the field for which they were designed, but as needing to be supplemented in the larger field of modern artillery fire, for which they were never intended.

QUESTIONS ON CHAPTER VI.

1. Give the dates and characteristics of the three periods of ballistic development.
2. What have been the two great obstacles to the development of exterior ballistics?
3. Define the three branches of ballistics.
4. Compare the present assumption relative to atmospheric density with that of the preceding period.
5. Compare the treatment of the equations of motion.
6. Compare the assumptions relative to rigidity.
7. Compare the assumptions relative to atmospheric resistance.
8. Compare the treatment of the ballistic coefficient.
9. Compare the coordinate systems.
10. What new corrections have been introduced by present methods?
11. Into what three parts does the study of exterior ballistics naturally divide?

CHAPTER VII.

THE MOTION OF A PROJECTILE.

For the purposes of the computation of trajectories and differential corrections, the projectile in flight is treated as a particle. The motion may thus be confined to the plane of fire, and the various effects of the oblique presentation of the projectile to the air (i. e., drift and the range effects of yaw) may be temporarily disregarded. Drift is subsequently treated as an empirical deflection correction; and the range effect of yaw will probably be treated as a differential range correction when the gyroscopic action of the projectile has been studied somewhat further than at present.

The atmospheric retardation of a projectile in flight in still air depends upon three things, viz., the velocity of the projectile, its physical characteristics, and the density of the air. Then the atmospheric acceleration (i. e., the negative of the retardation) can be expressed as follows:

26. $$\alpha = -\frac{FH}{C}$$

where F is a tabular empirical function of v. This is the F of the Siacci ballistics, and is not to be confused with the E (formerly represented by F) of present ballistic methods. H represents the actual density in terms of standard density, and C is an empirical constant (different for each projectile), employed to make the acceleration correspond to the physical characteristics of the projectile.

By throwing all the effects of a change in velocity into F, then H and C can be made independent of velocity. By regarding the atmospheric density at the gun as constant, H becomes a function of y (i. e., the altitude of the projectile above the level of the gun). The exponential function ($H = 10^{-0.000045y}$) has been found to be a close enough approximation to the physical facts for all practical purposes. C is conceived of as an empirical function of the characteristics of the projectile, and is constant over any given trajectory.

For standard density, H becomes unity. For the so-called standard projectile, C is unity.[1] Thus F is the acceleration of a standard projectile, traveling through standard air, at velocity v.

[1] This sentence is to be taken as the *definition* of "standard projectile." A standard projectile may be of any form, weight, etc., provided only that its C is unity.

For convenience (as will later appear), F is replaced by vG, G thus being the ratio of retardation to velocity of a standard projectile traveling through standard air at velocity v. For further convenience, the argument of the G tables is $\frac{v^2}{100}$.

Now it has been found by observation that the true G is practically dependent upon velocity alone, regardless of C, H, and θ. The very slight effects of changes in C, H, and θ on the true G have therefore been taken out of G, have been treated as constant throughout any given trajectory, and have been merged in the C of that trajectory, leaving the G function a function of v alone.

Equation 26 becomes:

27. $$\alpha = -\frac{FH}{C} = -\frac{GH}{C}v = -Ev$$

E being adopted merely as a convenient expression for $\frac{GH}{C}$.

The curve obtained by plotting G against v, shows that this function changes most rapidly around the velocity of sound ($v = 330$ m/s) and apparently has an inflection at that point. This suggests a relation between this function and the velocity of sound, which leads to expressing G as vB, where B is a function of the ratio between v and the velocity of sound. This relation is not accidental. It has a physical basis which can be derived theoretically.

This gives G the dimensions of velocity. H has the dimensions of density. Accordingly C has the dimensions of sectional density, i. e., weight divided by length squared.[1] This suggests representing C as $\frac{w}{d^2}$, where w is the weight of the projectile and d its diameter. But to make theory correspond to observation, it is necessary to insert an empirical factor of ignorance (i) in the denominator, hence $C = \frac{w}{id^2}$.

Let us now resolve the α of equation 27 into horizontal and vertical components, as follows:

28. $$\begin{cases} x'' = -Ev\cos\theta = -Ex' \\ y'' = -Ev\sin\theta = -Ey' \end{cases}$$

But there is an additional impressed acceleration, namely, gravity ($-g$). As this acts only vertically, equations 28 become:

29. $$\begin{cases} x'' = -Ex' \\ y'' = -Ey' - g \\ v^2 = x'^2 + y'^2 \\ \tan\theta = \frac{y'}{x'} \end{cases}$$

[1] Any other dimensions might be chosen for G, H, and C, provided that they combine to give correct dimensions to E. See supplement C.

These are the equations of motion of a projectile at any point of its flight, referred to cartesian axes horizontal and vertical, respectively, at that point.

But the earth is not flat. Accordingly, cartesian axes, which are horizontal and vertical at *one* point on the earth's surface, will not be so at any other. Furthermore, gravity decreases with altitude. The question therefore arises, how to adapt equations 29 to the actual state of the earth.

The most convenient ways would be as follows:

(*a*) "The tangent method." Consider the x axis as horizontal at the gun. The equations become (see supplement E):

$$\textbf{30.}\qquad \begin{cases} x'' = -Ex' - \dfrac{xg_0}{R} + \cdots \\ y'' = -Ey' - g_0 + \dfrac{2\,yg_0}{R} + \cdots \\ v^2 = x'^2 + y'^2 \\ \tan\theta = \dfrac{y'}{x'} \end{cases}$$

Where R is the radius of the earth, and g_0 is 9.80 meters per second-squared. All ranges, altitudes, and slopes are relative to the axes rather than to the earth, and the range at least must be corrected so as to relate to the earth, even in the case of short-range fire. But for purposes of computation, equations 29 are a sufficient approximation to equations 30, except in the case of long-range fire.

(*b*) "The curved method." Measure x in a circle passing through the gun and concentric with the earth. Measure y vertically from this circle. The equations become (see supplement E):

$$\textbf{31.}\qquad \begin{cases} x'' = -Ex' - \dfrac{2\,x'y'}{R} + \cdots \\ y'' = -Ey' - g_0 + \dfrac{2\,yg_0}{R} + \dfrac{x'^2}{R} + \cdots \\ v^2 = x'^2 + y'^2 + \dfrac{2\,yx'^2}{R} + \cdots \\ \tan\theta = \dfrac{y'}{x'}\left(1 - \dfrac{y}{R} + \cdots\right) \end{cases}$$

All ranges, altitudes, and slopes relate to the earth. For purposes of computation, equations 29 are a sufficiently close approximation to equation 31, except in the case of extremely long-range fire.

(*c*) Trajectories could also be computed by a third method which treats the x axis as tangent to the trajectory at the summit. This is sometimes called "the secant method," for it makes the x line through the gun secant to the earth.

All range tables now computed are based on trajectories computed by "the uncorrected curved method," i. e., x is measured along the surface of the earth and y is measured vertically from the surface of the earth, but equations 29 are used as an approximation to the more precise equations 31. The reason for adopting the curved convention is expressed as follows in the Aberdeen instructions for range firing:

"The level surface may be taken to be either the curved surface of the earth or the tangent plane to the earth at the position of the gun. For the purpose of the gunner, the former would be the more convenient in case the levels of gun and target were taken from a contour map, and the latter in case the levels were determined by sighting from the gun. As the difference would be insignificant except at long ranges, where presumably the first method would be employed, it is the custom at Aberdeen to determine the range for the curved surface."

Questions on Chapter VII.

1. Why is the projectile treated as a particle?
2. Upon what three things does atmospheric retardation depend?
3. What is the "standard projectile"?
4. What are the dimensions of E, H, G, and C?
5. What are the relative advantages of the "tangent method" and the "curved method"?
6. How are the x and y of a modern trajectory computation considered to be measured?
7. What are the equations of motion used in computing such a trajectory?

CHAPTER VIII.

COMPUTATION OF TRAJECTORIES.

[Rectangular method.]

In modern methods, a standard trajectory is computed from given values of ϕ, V, and C, by means of numerical integration (see Chap. IV) and successive approximations (see Chap. II).

The following tables are used:

Logs and $\frac{v^2}{100}$

Table of the G Function.

$\text{Log}_{10} H = -0.000045y$.

Some computers consider it quicker to compute $\log_{10} H$ by subtracting $\frac{y}{20}$ from $\frac{y}{2}$, pointing off four places, and then subtracting from zero, than it is to use the last mentioned table; accordingly the H table may be omitted, if desired.

The following blank forms are used:[1]

"Trajectory sheet," Form 5042.

"Computing sheet," Form 5041.

x, x', x'', y, y,$'$ and y'' computed as follows:

For each of these variables there are, on the trajectory sheet, four blank columns, the left-hand column being for the variable itself and the other three columns being for the first, second, and third differences, respectively.

The initial data are: The muzzle velocity (V) in meters per second, the ballistic coefficient (C), and the angle of projection (ϕ). On the first line of the trajectory sheet enter the initial values of x, x', t, y, and y', as follows:

$$x = 0$$
$$x' = V \cos \phi$$
$$t = 0$$
$$y = 0$$
$$y' = V \sin \phi$$

Then turn to the small computing sheet and compute the first column, for $t = 0$. On this sheet, $\log x'^2$ and $\log y'^2$ should be crossed out and $\log v'$ (which is a misprint) changed to $\log y'$. Colog C and g are constants throughout the computation. Colog C is obtained

[1] Alternative methods of much merit are given in Supplements A and G. Of the three methods, that of Supplement G is the one at present favored by the computers of the Technical Staff.

from the initial data; g is 9.80. On both sheets F should be changed to E, to conform to the approved notation. The logarithms are all to base, 10.

The computing sheet is used to obtain x'' and y'' from the fundamental equations of motion, derived in the preceding chapter:

32. $$\begin{cases} x'' = -Ex' \\ y'' = -Ey' - g \\ E = \dfrac{GH}{C} \end{cases}$$

The process is as follows: Taking the value of x' from the trajectory sheet, look up at the same time log x' and $\frac{x'^2}{100}$ in the table of logs and squares. Similarly look up log y' and $\frac{y'^2}{100}$.

Add $\frac{x'^2}{100}$ and $\frac{y'^2}{100}$ to get $\frac{v^2}{100}$. With the latter, enter the G table, and take out log G.

With y, as tabulated on the trajectory sheet, enter the H table and take out log H; or compute log H from y without using the table.

Add log G, log H, and colog C to get log E.

Add log x' and log E to get log Ex'. Add log E and log y' to get log Ey'.

Look up Ex' and Ey' from their logs, in the table of logs and squares. Be careful to give Ex' the same sign as x' and Ey' the same sign as y'.

Add Ey' and g algebraically, to get $Ey'+g$.

Change the sign of Ex' and of $Ey'+g$, and enter them on the trajectory sheet on the same line as the data on which they were based. The reason for this change of sign is that x'' equals minus Ex' and not plus Ex'. Similarly for y''.

We are now ready to begin a new line on the trajectory sheet. Our second t should usually be $\frac{1}{4}$ second. Enter this value in the "Time" column. Two formulas for integration ahead are available for breaking into a new line. This first formula is:

33. $$\int_t^{t+i} f\,dt = i\left(f_t + \frac{1}{2}a_t + \frac{5}{12}b_t + \frac{3}{8}c_t + \frac{1}{3}d_t\right)$$

where t is the time of the last complete line and i is the time interval. For example, if f_t represents the tabulated value of x'' at time t, then the integral represents the increment of x' during the interval from t to $t+i$.

In proceeding from $t=0$ to $t=\frac{1}{4}$, we have no values of a, b, c, etc., corresponding to x'', and so have only the rough approximation:

$$\int_0^{\frac{1}{4}} x''\,dt = \frac{1}{4}\,x_0''.$$

As this integral represents the increment of x' from time 0 to time $\frac{1}{4}$, its value should be entered in the a column corresponding to x' for time $\frac{1}{4}$. Add algebraically to the value of x' for time 0, and enter the sum as the tentative value of x' for time $\frac{1}{4}$.

Similarly, integrate ahead for a tentative value of y' for $\frac{1}{4}$.

We have now broken into our new line, and are in a position to get a tentative value for y for time $\frac{1}{4}$, by means of the standard integration formula

34. $$\int_{t-i}^{t} f\,dt = i\left(f_t - \frac{1}{2}a_t - \frac{1}{12}b_t - \frac{1}{24}c_t - \frac{1}{40}d_t\right)$$

Now return to the small sheet, and compute a new column, as before, using the tentative values of x', y', and y for $t=\frac{1}{4}$. Enter the resulting values of x'' and y'' on the trajectory sheet on the line for $t=\frac{1}{4}$, and integrate (this time by formula 34) for x', y', and y. Substitute these improved values for the tentative values previously set down, and repeat the computation on the small sheet.

Continue this process of successive approximations until you obtain values of x', x'', y, y', and y'', which check throughout. Then integrate for x by formula 34. The smoothness of the differences of x is a valuable check on the accuracy of x'.

Then proceed, in the same way, to get the values of x', x'', y, y', y'', and x, for $t=\frac{1}{2}$, $t=\frac{3}{4}$, and $t=1$. Each succeeding line will, of course, furnish more of a, b, c, etc., for use in formulas 33 and 34.

After completing the line for $t=1$, it may be well, though not essential, to extrapolate back for values for a, b, and c for $t=0$, and repeat the computation up to $t=1$ again. This results in smoothing out the curve and giving greater precision to the initial steps. By using this smoothing-out process, it is often possible to start with a one-second interval[2] and obtain as precise results as a quarter-second interval would give without the smoothing-out process.

After completing the line for $t=1\frac{1}{2}$, skip a few lines, copy down the values for x, x', x'', y, y', and y'' for times 0, $\frac{1}{2}$, 1, and $1\frac{1}{2}$, and make up new columns of a and b corresponding to each of these, but now based on a *half*-second interval instead of on a *quarter*-second interval as before.

Whenever third differences are available, the following formula for integrating ahead will be found much simpler than formula 33:

35. $$\int_{t-i}^{t+i} f dt = i\,[2f_t + \tfrac{1}{3}\,(b_t + c_t + d_t + \cdot\cdot\cdot)\,]$$

It is to be noted that the *first* difference (a_t) does not enter into this formula, and that the increment obtained is to be added to the value for the *preceding* line. For example, in integrating x'' at time

[2] See Supplement G for a method of starting at one-second interval.

20 to get x' for time 21, add the increment obtained by formula 35 to the value of x' for time 19. The increment, if obtained by formula 33, would be added to the value of x' for time 20. When first using formula 35, it would be well to use formula 33 too, and compare results, as a check.

Go on from this point at half-second intervals, remembering that i now equals $\frac{1}{2}$. Continue the process until y becomes negative. This means that the projectile has pierced the initial plane and has passed below the point of fall.

It is well to carry x'' and y'' to hundredths, and x', x, y', and y to tenths of meters throughout the computation.

Often during the computation, and certainly at its close, the total increment of x, x', y, and y', from time zero, should be checked by integrating x', x'', y', and y'' by Simpson's rule, preferably using some type of calculating machine.

Usually, in computing trajectories, a one-second interval is adopted shortly after the start, the change of interval being accomplished in a manner similar to that prescribed for changing to a half-second interval.

Intervals greater than one second should not be used by the student, for the reason that data at one-second intervals are frequently necessary for the subsequent computation of the differential corrections.[3]

To get the terminal values of t, x, x', and y' (i. e., the values corresponding to $y=0$), it was formerly the practice to use formula 6, and the interpolation formula of problem 30. The following somewhat analogous formulas[4] are simpler:

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[3] The present practice of the Technical Staff (whether the method of this Chapter or that of Supplement G is employed) is to double the interval every time the third differences in the doubled interval indicate that fourth differences may be neglected. Accordingly, only very short trajectories are computed throughout by a one-second interval, the usual intervals at the end being two or four seconds. The differential corrections are seldom required to any intermediate time. When they are so required, it is less work for the skilled computer to find them by interpolation than by retaining a smaller interval in the trajectory.

[4] Taylor's expansion of $y_{t+\Delta_t}=0$ may be written strictly:

$$-y_t=y_{t+\Delta_t}-y_t=\Delta t\left\{y'_t+\frac{\Delta t}{2}\left[y''_t+\frac{\Delta t}{3}\left(y'''_t+\ \ldots\ .\right)\right]\right\}.$$

Similarly:

$$\Delta y'=y'_{t+\Delta_t}-y'_t=\Delta t\left\{y''_t+\frac{\Delta t}{2}\left[y'''_t+\frac{\Delta t}{3}\left(y^{iv}_t+\ \ldots\ .\right)\right]\right\}.$$

In the approximate formulas of the text, all terms in y''_t and higher derivatives have been dropped. To get a closer approximation, make use of:

$$\Delta y''\approx y'''_t\Delta t$$

$$y'''_t\approx\frac{(a\text{ of }y''_t)}{i}$$

$$y''_t+\frac{\Delta t}{3}y'''_t\approx y''_t+\frac{\Delta t}{2}y'''_t.$$

This gives us:

$$\Delta y''=\frac{(a\text{ of }y''_t)}{i}\Delta t$$

$$\Delta y'=\left(y''_t+\frac{\Delta y''}{2}\right)\Delta t$$

and similarly for $\Delta x''$ and $\Delta x'$, the other formulas all remaining the same.

36. $$\Delta t = \frac{-y_t}{y'_t + \frac{\Delta y'}{2}}; \quad \Delta y' = y''_t \Delta t.$$

Note that the Δt of this method equals $i\Delta t$ of formula 6. The other deltas are the same in both methods.

Take the values of x, x', x'', t, y, y', and y'' from the line in which y has the smallest absolute values, and keep a strict account of algebraic signs. Solve for Δt by successive approximations,[5] taking as a first approximation:

$$\Delta t = \frac{-y_t}{y'_t}.$$

When Δt has been found, the terminal values are obtained as follows:

37. $$\begin{cases} \Delta x' = x''_t \Delta t \\ \Delta x = \left(x'_t + \frac{\Delta x'}{2}\right) \Delta t \\ T = t + \Delta t \\ y'_T = y'_t + \Delta y' \end{cases}$$

Skip a few lines, and tabulate the values corresponding to $t = T$. Compute the angle of fall by the formula:

$$\tan \omega = -\frac{y'_T}{x'_T}$$

PROBLEMS.

(46) Compute the standard trajectory which would result from the following initial conditions:

$V = 792.47$ meters per second.
$C = 4.0$
$\varphi = 2°$

[5] The work on a slide rule is as follows:
(a) Set the runner to y_t on scale D.
(b) Bring y'_t on scale C up to the runner.
(c) Opposite 1 on scale D, read the approximate Δt on scale C, and jot it down for a check.
(d) Opposite (a of y''_t) on scale C, read the approximate $\Delta y''$ on scale D.
(e) Opposite $y''_t + \frac{\Delta y''}{2}$ on scale C, read the approximate $\Delta y'$ on scale D.
(f) Bring $y'_t + \frac{\Delta y'}{2}$ on scale C up to the runner.
(g) Opposite 1 on scale D, read a new approximate Δt on scale C, compare with one found above.
(h) Repeat the process from (d) to (g), until no further shift of the slide is necessary. Algebraic signs must always be taken into consideration. Also, be careful to see that the various figures are correctly pointed off, so that, for example, a Δt that should be .043 is not used as .43.

Then Δt, $\Delta y''$, $\Delta y'$, y_t, (as a check), $\Delta x''$, $\Delta x'$, and Δx can be severally read on scale D without change of slide. They will be respectively opposite 1, (a of x''_t)/i, $y''_t + \frac{\Delta y''}{2}$, $y'_t + \frac{\Delta y'}{2}$, ($a$ of x''_t)/i, $x''_t + \frac{\Delta x''}{2}$, and $x'_t + \frac{\Delta x'}{2}$.

Start with quarter-second intervals. Do not use the smoothing-out process described on page 49. Change to half-seconds after $t=1\frac{1}{2}$, and to seconds after computing $t=3$.

(47) Use the same initial conditions and half-second intervals. After computing the line for $t=2$, extrapolate back for a, b, and c of x'' and y'', for $t=0$, and recompute to $t=2$. Change to one-second intervals at $t=3$. Compare the results with problem 46.

Questions on Chapter VIII.

1. What two mathematical processes are used in computing a standard trajectory?
2. What tables are used?
3. Briefly, just what is a "standard trajectory"?
4. What initial data are used?
5. What equations of motion are used?
6. What are the formulas for integration ahead?
7. When is each used?
8. What is the standard integration formula?
9. How are the computations checked from time to time?

CHAPTER IX.

DERIVATION OF AUXILIARY VARIABLES.

A given ballistic coefficient, muzzle velocity, and angle of departure determine a standard trajectory, all the elements of which can be computed by the methods laid down in the preceding chapter. In that chapter we saw that any given trajectory has, for each value of t, a definite value for x, y, x', y', x'', and y''. The object of the present chapter is to derive three auxiliary variables, μ, ν, and ρ, which shall each have a definite value for each point on each given trajectory. In the next chapter we shall see how the formulas for the various range corrections can be expressed in terms of these auxiliary variables.

In Chapter III we saw how the effect on x at time T, due to a disturbance at time t_Δ, could be expressed as in equation 17:

$$\delta X = L\ \delta x + M\ \delta y + N\ \delta x' + P\ \delta y'$$

This is a general expression relating to the motion of any particle moving under any definite law and subject to a disturbance at one instant of time. Let us now specialize this expression, as follows: Let the particle be a projectile moving in accordance with the laws of motion evolved in Chapter VII. Let T represent the time of the point of fall of the standard undisturbed trajectory. Then X is the standard range. Let λ, $h\mu$, ν, and ρ respectively be L, M, N, and P, specialized by these conditions. h is the h of the exponential expression for H (i. e., $H = e^{-hy}$).

λ is thus the change in range due to a unit change in x at time t_Δ; $h\mu$ is the change in range due to a unit change in y at time t_Δ, etc. Thus these variable conversion factors are seen to have an important physical significance. It is obvious that since x does not enter into the differential equations of motion of a projectile, a change in x at any point can have no effect other than to shift the value of x at all subsequent points by the same amount. Therefore λ equals unity, and the expression for a range change may be written:

38. $$\delta X = \delta x + h\mu\ \delta y + \nu\ \delta x' + \rho\ \delta y'$$

The effect of acceleration changes will not be considered in this chapter.

Strictly speaking, the new range (i. e., $X + \delta X$) will not be the range to the point of fall of the disturbed trajectory (i. e., the point where

$y=0$), but rather will be the range to the point on the disturbed trajectory where $t=T$. The time of the point of fall on the disturbed trajectory will be $T+\delta T$, where δT is the change in T due to the disturbance. The x effect, at time T, of the disturbance at time t_Δ will differ from the x effect at time $T+\delta T$ by some proportionally very small part of the very small quantity δX, and hence this very small part of δX may be disregarded, by theorem 9.

In equation 38, δX is independent of t_Δ, by theorem 18. By theorem 15:

$$\frac{d}{dt_\Delta}(\delta u)=\delta\left(\frac{du}{dt}\right)=\delta u'$$

Therefore, differentiating equation 38 with respect to t_Δ, we get:

$$\textbf{39.}\quad 0=\delta x'+h\frac{d\mu}{dt_\Delta}\delta y+h\mu\ \delta y'+\frac{d\nu}{dt_\Delta}\delta x'+\nu\ \delta x''+\frac{d\rho}{dt_\Delta}\delta y'+\rho\ \delta y''$$

Now, by theorem 14, since x'' and y'' are each a function of x', y', and y only:

$$\textbf{40.}\quad \begin{cases} \delta x''=\dfrac{\partial x''}{\partial x'}\delta x'+\dfrac{\partial x''}{\partial y'}\delta y'+\dfrac{\partial x''}{\partial y}\delta y \\ \delta y''=\dfrac{\partial y''}{\partial x'}\delta x'+\dfrac{\partial y''}{\partial y'}\delta y'+\dfrac{\partial y''}{\partial y}\delta y \end{cases}$$

Let us designate $\dfrac{d\mu}{dt_\Delta}$ by μ', and similarly for ν' and ρ'.

Substituting the values from equations 40 in equation 39 and collecting the terms, we get:

$$\textbf{41.}\quad 0=\left(h\mu'+\nu\frac{\partial x''}{\partial y}+\rho\frac{\partial y''}{\partial y}\right)\delta y+\left(1+\nu'+\nu\frac{\partial x''}{\partial x'}+\rho\frac{\partial y''}{\partial x'}\right)\delta x'+$$
$$\left(h\mu+\rho'+\nu\frac{\partial x''}{\partial y'}+\rho\frac{\partial y''}{\partial y'}\right)\delta y'.$$

Now since δy, $\delta x'$, and $\delta y'$ were chosen independently at time t_Δ the algebraic law of vanishing coefficients applies, and each parenthesis above equals zero, whence:

$$\textbf{42.}\quad \begin{cases} h\mu'=-\nu\dfrac{\partial x''}{\partial y}-\rho\dfrac{\partial y''}{\partial y} \\ \nu'=-1-\nu\dfrac{\partial x''}{\partial x'}-\rho\dfrac{\partial y''}{\partial x'} \\ \rho'=-h\mu-\nu\dfrac{\partial x''}{\partial y'}-\rho\dfrac{\partial y''}{\partial y'} \end{cases}$$

Let us now perform the partial differentiations indicated in equations 42. Since

$$x''=-Ex', \text{ and}$$

$$y''=-Ey'-g,$$

then,

43.
$$\begin{cases} \dfrac{\partial x''}{\partial x'}=-E-x'\dfrac{\partial E}{\partial x'} \\ \dfrac{\partial x''}{\partial y}=-x'\dfrac{\partial E}{\partial y} \\ \dfrac{\partial x''}{\partial y'}=-x'\dfrac{\partial E}{\partial y'} \\ \dfrac{\partial y''}{\partial x'}=-y'\dfrac{\partial E}{\partial x'} \\ \dfrac{\partial y''}{\partial y}=-y'\dfrac{\partial E}{\partial y} \\ \dfrac{\partial y''}{\partial y'}=-E-y'\dfrac{\partial E}{\partial y'} \end{cases}$$

Let us now perform the partial differentiations of E, indicated in equations 43. Since

$$E=\frac{GH}{C}, \text{ and}$$

$$v^2=x'^2+y'^2, \text{ and}$$

$$H=e^{-hy},$$

then,

44.
$$\begin{cases} \dfrac{\partial E}{\partial x'}=\dfrac{H}{C}\cdot\dfrac{\partial G}{\partial x'}=\dfrac{E}{G}\cdot\dfrac{dG}{dv}\cdot\dfrac{\partial v}{\partial x'}=Ex'\left(\dfrac{d\log G}{v\,dv}\right) \\ \dfrac{\partial E}{\partial y}=\dfrac{G}{C}\cdot\dfrac{\partial H}{\partial y}=\dfrac{E}{H}(-hH)=-hE \\ \dfrac{\partial E}{\partial y'}=\dfrac{H}{C}\cdot\dfrac{\partial G}{\partial y'}=\dfrac{E}{G}\cdot\dfrac{dG}{dv}\cdot\dfrac{\partial v}{\partial y'}=Ey'\left(\dfrac{d\log G}{v\,dv}\right) \end{cases}$$

Substituting the values from equations 43 and 44 in equations 42, we get:

45.
$$\begin{cases} \mu'=-E(x'\nu+y'\rho) \\ \nu'=-1+E\nu-\mu'x'\left(\dfrac{d\log G}{v\,dv}\right) \\ \rho'=-h\mu+E\rho-\mu'y'\left(\dfrac{d\log G}{v\,dv}\right) \end{cases}$$

These equations hold for any possible combination of δx, δy, $\delta x'$, and $\delta y'$ occurring at any point on the trajectory.

In these equations, the auxiliary variables are, by their original definition, functions of the time of disturbance, t_Δ. The elements E, x', y', G, and V are functions of t in the original trajectory computation; but it is evident that their values, for insertion in equations 45, must be the values which they have at the point of disturbance, i. e., at the point $t = t_\Delta$. This consideration becomes important in the solution of these equations in the next chapter.

Equations 45 can be solved for any given trajectory by making use of the terminal values of μ, ν, and ρ, which are as follows:

$$\text{46.} \qquad \begin{cases} \mu_T = \dfrac{\cot \omega}{h} \\ \nu_T = 0 \\ \rho_T = 0 \end{cases}$$

These values are true because, at the point of fall a unit increase in y produces an increase in X equal to $\cot \omega$, but a unit change in x' or y' produces no change in X.

Auxiliary variables may in a similar manner be derived with respect to time of flight (T) and maximum ordinate (y_s). The three sets are then distinguished by the subscripts 1, 2, and 3.

$$\delta X = \lambda_1 \delta x + h\mu_1 \delta y + \nu_1 \delta x' + \rho_1 \delta y'$$

$$\delta T = \lambda_2 \delta x + h\mu_2 \delta y + \nu_2 \delta x' + \rho_2 \delta y'$$

$$\delta y_s = \lambda_3 \delta x + h\mu_3 \delta y + \nu_3 \delta x' + \rho_3 \delta y'$$

Hereafter in this book (except in supplement D) the omission of subscripts signifies that the auxiliary variables relate to *range* changes.

PROBLEMS.

(48) What are the formulas for λ'_2, μ'_2, ν'_2, and ρ'_2?

(49) What initial, terminal, and constant values are there for λ_2, λ'_2, etc.?

(50) What are the formulas for λ'_3, μ'_3, ν'_3 and ρ'_3?

(51) What initial, terminal, and constant values are there for λ_3, λ'_3, etc.? (Note that here the terminus is the summit.)

The computation of differential corrections has been simplified by the discovery of the following so-called "new first integral":

$$\text{47.} \qquad x' + h\mu_1 y' + \nu_1 x'' + \rho_1 y'' = 0$$

This may be derived as follows. Consider equation 38:

$$\delta x + h\mu_1 \delta y + \nu_1 \delta x' + \rho_1 \delta y' = \delta X$$

Restrict the four independent variables (i. e., δx, δy, $\delta x'$, and $\delta y'$) by the condition that they shall represent a change from point to point on the same trajectory, in the same infinitesimal time interval, dt. This converts the operator δ to the operator d, and makes the range change equal to zero. Thus:

$$dx + h\mu_1 dy + \nu_1 dx' + \rho_1 dy' = 0$$

Divide through by dt. Then:

48. $$x' + h\mu_1 y' + \nu_1 x'' + \rho_1 y'' = 0$$

The similar "first integral" for $\frac{dT}{dt}$, equals unity; for $\frac{dy_s}{dt}$ it equals zero.

The above derivation gives the following physical meaning to the "first integrals," namely, that an infinitesimal change in the four elements (x, y, x', and y') in time dt, which change amounts to shifting from one point to another of the *same* trajectory, has no effect on range or maximum ordinate, and makes a *proportional* change in the time of flight.

Now substitute $-Ex'$ for x'' and $-Ey' - g$ for y'' in equations 47. The "first integral" thus becomes:

$$x' + h\mu_1 y' - E\,(x'\nu_1 + y'\rho_1) - g\rho_1 = 0$$

or:

49. $$x' + h\mu_1 y' + \mu_1' = g\rho_1$$

whence:

$$\rho_1 = \frac{x' + h\mu_1 y' + \mu_1'}{g}$$

Differentiating:

$$\rho_1' = \frac{x'' + h\mu_1 y'' + h\mu_1' y' + \mu_1''}{g}$$

Substituting these values in the last one of equations 45, namely, in:

$$\rho' = -h\mu_1 + E\rho_1 - \mu'_1 y' \left(\frac{d \log G}{v\, dv}\right)$$

we get:

$$x'' + h\mu_1 y'' + h\mu_1' y' + \mu''_1 = -gh\mu_1 + E(x' + h\mu_1 y' + \mu'_1) - g\mu_1' y' \left(\frac{d \log G}{v\, dv}\right),$$

whence comes the following equation:

$$\mu_1'' = 2hEy'\mu_1 + \left[E - hy' - gy'\left(\frac{d \log G}{v\, dv}\right)\right]\mu'_1 + 2\,Ex',$$

which is the equation used in computing the auxiliary variables. This equation produces the value of μ_1' and μ_1 for each value of t_Δ. The values of ρ_1 and ν_1 can then be found from the formulas:

$$\rho_1 = \frac{1}{g}(x' + hy'\mu_1 + \mu_1'),$$

$$\nu_1 = -\frac{y'}{x'}\rho_1 - \frac{\mu'_1}{Ex'}$$

The former of these two is derived from equation 47. The latter is obtained from the first of equations 45.

Thus we have, as the three equations of the system:

$$\text{50.} \quad \begin{cases} \mu''_1 = 2hEy'\mu_1 + \left[E - hy' - gy'\left(\dfrac{d \log G}{v\, dv}\right)\right] \mu'_1 + 2\ Ex' \\ \rho_1 = \dfrac{1}{g}\ (x' + hy'\mu_1 + \mu'_1) \\ \nu_1 = -\dfrac{y'}{x'}\rho_1 - \dfrac{\mu'_1}{Ex'} \end{cases}$$

The method of solving these equations is given in Chapter XIV.

PROBLEM.

(52) Derive the corresponding equations for μ_2'', ρ_2, and ν_2; and μ_3'', ρ_3, and ν_3.

Questions on Chapter IX.

1. Why does the "first integral" $\frac{dT}{dt}$ equal unity; and the "first integral" $\frac{dy_t}{dt}$ equal zero?
2. What is the physical interpretation of λ_1, $h\mu_1$, μ_1, and ρ_1?
3. What is the physical interpretation of the fact that λ_1 equals unity?
4. In this book what does the omission of a subscript from the auxiliary variables signify?
5. The auxiliary variables are functions of what, on any given trajectory?
6. Of what are x, y, x', y', etc., functions, on any given trajectory?
7. Is it correct to state that the values of x' and y' used in the computation of auxiliary variables are functions of t_Δ? Explain.
8. Upon what subjects of mathematics, discussed earlier in this book, is this chapter based?
9. Does $X+\delta X$ equal the range of the disturbed trajectory? Explain.
10. Why are the terminal values of μ, ν, and ρ as given?

CHAPTER X.

RANGE CORRECTION FORMULAS.

IN GENERAL.

We have seen in the preceding chapter that at each point (t_Δ) on any standard trajectory there is a unique value for each of the auxiliary variables: λ_1, μ_1, ν_1, and ρ_1.

λ_1 represents the change in range caused by a unit variation in x, and is a constant. $\lambda_1 = 1$.

$h\mu_1$ represents the change in range caused by a unit variation in y.

ν_1 represents the change in range caused by a unit variation in x'.

ρ_1 represents the change in range caused by a unit variation in y'.

As only range changes are considered in this chapter, the subscripts will now be omitted, to simplify typesetting.

The total range change caused by disturbances may be expressed by consolidating all the range effects of equations 38 and 21 into one range effect, δX.

51. $$\delta X = [\delta x + h\mu\ \delta y + \nu\ \delta x' + \rho\ \ \delta y']_{t_0} + \int_{t_0}^{T} d\delta x + h\int_{t_0}^{T} \mu\, d\delta y + \int_{t_0}^{T} \nu\, d\delta x' + \int_{t_0}^{T} \rho\ d\delta y' + \int_{t_0}^{T} \nu\ \delta x'' dt_\Delta + \int_{t_0}^{T} \rho\ \delta y'' dt_\Delta$$

where the t_0 of any of the four bracketed terms is the time of the instantaneous disturbance considered in that term, and the t_0 of any of the six integral terms is the time of commencement of the disturbance considered in that term, and the t_0 of no two terms need necessarily be the same.

Any specialized range change will be represented as ΔX, with some subscript, so as to conform to range-table usage.

In the formulas of this chapter the auxiliary variables μ, ν, ρ, and their derivatives are functions of the time of disturbance, t_Δ. The proper value of x', y', E, G, etc. (which are normally functions of t), to insert in these formulas are the values for $t = t_\Delta$.

The formulas given in this book are all (with the exception of some in Chapter XI) formulas for the *increment* in range due to non-standard conditions. Therefore, to correct the observed range to standard range, or the map range to range-setter range, the quantity obtained from the proper formula must be algebraically *subtracted*. This is in conformity with an agreement between the technical staff of the Ordnance Department and the chiefs of Coast Artillery and

Field Artillery that range tables shall tabulate the range *effects of*, rather than the range *corrections for*, all nonstandard conditions other than height of site.

EFFECT OF NONSTANDARD MUZZLE VELOCITY.

At the muzzle,

$$x' = V \cos \phi$$
$$y' = V \sin \phi$$

A 1 *m/s* increase in V increases x' by $\cos \phi$, and increases y' by $\sin \phi$. Therefore, for $\delta V = 1$ *m/s:*

$$\delta x' = \cos \phi$$
$$\delta y' = \sin \phi$$

Therefore, from equation 51:

52. $$\Delta X_V = (\nu \cos \phi + \rho \sin \phi)_{t_\Delta = 0}$$

PROBLEM.

(53) Derive the alternative form:

$$\Delta X_V = -\left(\frac{\mu' \cos \phi}{Ex'}\right)_{t_\Delta = 0}$$

from the last equation of equations 49.

EFFECT OF CHANGE IN ϕ.

By problem 22, $\delta \cos \phi$ equals $-\sin \phi\, \delta\phi$, and $\delta \sin \phi$ equals $\cos \phi\, \delta\phi$. It is customary to base the correction on a 1-mil change in ϕ.

$$\delta\phi = 1 \text{ mil} = \frac{1}{1019} \text{ radians.}$$

A 1-mil increase in ϕ decreases x' by $V \dfrac{\sin \phi}{1019}$, and increases y' by $V \dfrac{\cos \phi}{1019}$. Therefore, for $\Delta\phi = 1$ mil.:

$$\delta x' = -V \frac{\sin \phi}{1019}$$
$$\delta y' = V \frac{\cos \phi}{1019}$$

Therefore, from equation 51:

53. $$\Delta X_\phi = \frac{V}{1019} (\rho \cos \phi - \nu \sin \phi)_{t_\Delta = 0}$$

PROBLEM.

(54) Derive the alternative form:

$$\Delta X_\phi = \frac{V}{1019}\left(\frac{\rho}{\cos\phi} + \frac{\mu' \sin\phi}{Ex'}\right)_{t_\Delta = 0}$$

CHANGES IN ACCELERATION, IN GENERAL.

From formula 51, the range effect of variations in acceleration is seen to be:

54. $$\delta X = \int_{t_0}^{T} \nu\, \delta x'' dt_\Delta + \int_{t_0}^{T} \rho\, \delta y'' dt_\Delta$$

We are interested only in such acceleration changes as are caused by variations in E, inasmuch as variations in x' or y' (the other factor of x'' and y'', respectively) are cared for by the terms in $\delta x'$ and $\delta y'$ in formula 51. Thus for the purposes of the present derivation we may assume that x' and y' do not vary. Accordingly:

55. $$\begin{cases} \delta x'' = \dfrac{\partial x''}{\partial E}\delta E = -x'\delta E = -x'E\left(-\dfrac{\delta C}{C} + \dfrac{\delta G}{G} + \dfrac{\delta H}{H}\right) \\ \delta y'' = \dfrac{\partial y''}{\partial E}\delta E = -y'\delta E = -y'E\left(-\dfrac{\delta C}{C} + \dfrac{\delta G}{G} + \dfrac{\delta H}{H}\right) \end{cases}$$

Substituting from equation 55 in equation 51:

$$\delta X = \int_{t_0}^{T} -\left(-\frac{\delta C}{C} + \frac{\delta G}{G} + \frac{\delta H}{H}\right) E(x'\nu + y'\rho)\, dt_\Delta$$

Substituting μ' for $-E(x'\nu + y'\rho)$, from equations 45, we get:

56. $$\delta X = \int_{t_0}^{T} \left(-\frac{\delta C}{C} + \frac{\delta G}{G} + \frac{\delta H}{H}\right) \mu' dt_\Delta$$

EFFECT OF NONSTANDARD BALLISTIC COEFFICIENT.

For a 10 per cent increase in C alone, formula 56 becomes:

$$\Delta X_C = \int_{t_0}^{T} (-0.1)\mu' dt_\Delta$$

57. $$\Delta X_c = 0.1[\mu_{t_0} - \mu_T]$$

EFFECT OF NONSTANDARD WEIGHT OF PROJECTILE.

An increase in the weight of the projectile over the standard weight will increase the ballistic coefficient (C) proportionally, and will decrease the muzzle velocity (V) in accordance with the formula of interior ballistics,

$$\frac{\delta V}{V} = n\frac{\delta p}{p}$$

where p is the standard weight of projectile, and n is one of the variable coefficients of interior ballistics (approximately -0.3), *not to be confused with the Gâvre n of the temperature formula* (equations 59 to 64).

A 1 per cent increase in the weight of the projectile will change the velocity by $0.01\ nV$, which in turn will change the range by that, multiplied into the range effect of a 1 m/s increase in V (formula 52). A 1 per cent increase in the weight of the projectile will also increase C by 1 per cent, which in turn will increase the range by one-tenth of the range effect of a 10 per cent increase in C (formula 57). Hence:

58. $$\Delta X_p = 0.1\ \Delta X_c + 0.01\ Vn\ \Delta X_v$$

For values of n, see lesson sheet P of the Ordnance School of Application: "Tables for Interior Ballistics," Table A; or use the value -0.3.

It might seem that n should equal -0.5 instead of -0.3; for if the muzzle energy produced by a given charge of powder were constant, regardless of the weight of the projectile, then:

$$\frac{1}{2}(p+\delta p)(V+\delta V)^2 = \frac{1}{2}pV^2$$

whence:

$$\frac{2\delta V}{V} = -\frac{\delta p}{p}$$

$$\frac{\delta V}{V} = -0.5\frac{\delta p}{p}$$

But a change in the weight of the projectile causes it to stay a different length of time in the bore and alters the friction between the gun and projectile, so that the muzzle energy is *not* constant. Empirically, -0.3 has been found to give more nearly correct results than -0.5.

In range-table firings, when the actual velocity is known, and what is desired is the standard range, one range correction is made (by formula 52) to correct from actual velocity to standard velocity, and one to correct from actual weight to standard weight, using only the first term of formula 58. The last term of formula 58 is not used, for formula 52 covers the whole matter of velocity.

If the velocities are taken on special velocity rounds, rather than on the range rounds, the velocity of the range rounds is obtained by adding δV to the velocity of the velocity rounds, δV being found by the formula:

$$\frac{\delta V}{V} = n\frac{\delta p}{p}$$

where δp is the difference obtained by subtracting the average projectile weight of the velocity rounds from that of the range rounds.

In service firings (or in any case where the range obtained by one weight of projectile is known, and what is desired is the range to be expected of another weight, all other things being unchanged) the whole of formula 58 should be used.

EFFECT OF NONSTANDARD TEMPERATURE.

A 1 per cent change in absolute temperature alone will now be considered, merely from the viewpoint of its effect on the elasticity of the air.

Let s represent the velocity of sound, s_0 the standard velocity of sound, τ the absolute temperature, and τ_0 the standard absolute temperature for the altitude in question. A normal temperature structure has been adopted by the technical staff of the Ordnance Department, based upon the normal temperature which occurs at each altitude up to ten thousand meters, when the temperature at the ground is standard. This temperature structure is incorporated in a table, giving

$$\frac{10^3}{2\tau_0}$$

with y as an argument. At the ground τ_0 equals 518.4° in Fahrenheit units, corresponding to 59° Fahrenheit.[1]

Represent G as the product of the velocity by a function of $\frac{v}{s}$. Call this function $B\left(\frac{v}{s}\right)$. Then:

$$G = v\,B\left(\frac{v}{s}\right)$$

$$G_0 = v\,B\left(\frac{v}{s_0}\right)$$

Let $s = s_0 + \delta s$. Then:

$$G = v\,B\left(\frac{v}{s_0 + \delta s}\right)$$

Expanding by Taylor's theorem:

$$G = v\,B\left(\frac{v}{s_0}\right) - v^2\,\frac{dB}{dv}\,\frac{\delta s}{s_0} + \text{higher powers of } \frac{\delta s}{s_0}$$

59. $$\delta G = G - G_0 = -v^2\,\frac{dB}{dv}\,\frac{\delta s}{s_0}\ \ldots$$

It now remains to evaluate $\frac{dB}{dv}$ and $\frac{\delta s}{s_0}$

[1] See "Physical Bases" (Ordnance Textbook 972), pp. 12, 13. Also note, p. 34, *ante*.

60. $$G = vB = \frac{F}{v}$$

(N. B.—This F is not to be confused with the E of modern ballistic methods, which n many of the earlier papers was written F.)

$$F = A_n v^n$$

$$G = A_n v^{n-1}$$

61. $$B = A_n v^{n-2}$$

Differentiating equation 61:[1]

62. $$\frac{dB}{dv} = \frac{d(A_n v^{n-2})}{dv} = (n-2)A_n v^{n-3} = (n-2)\frac{G}{v^2}.$$

Differentiating equation 60:

$$\frac{dB}{dv} = \frac{v\,dG - G\,dv}{v^2 dv} = \frac{G}{v^2}\left(\frac{v\,dG}{G\,dv} - 1\right) = \frac{G}{v^2}\left(\frac{v\,d\log G}{dv} - 1\right).$$

Therefore the Gâvre n may be defined by the equation:

$$n = \frac{v\,d\log G}{dv} + 1.$$

The object of the above differentiations was to get n into the form of a logarithmic derivative of G.

Now to evaluate $\frac{\delta s}{s_0}$. In the vicinity of s_0, s varies as $\sqrt{\tau}$. Therefore:

63. $$\frac{\delta s}{s_0} = \frac{\delta\tau}{2\tau_0}.$$

Substituting from equations 62 and 63 in equation 59:

$$\frac{\delta G}{G} = -\frac{v^2}{G}\frac{dB}{dv}\frac{\delta s}{s_0} = -(n-2)\frac{\delta\tau}{2\tau_0}.$$

A $\delta\tau$ of 1 per cent would make:

$$\frac{\delta G}{G} = -(n-2)\frac{1}{200}.$$

Substituting in equation 56:

64. $$\Delta X_T = -0.005\int_{t_0}^{T}(n-2)\mu' dt_\Delta.$$

[1] A_n and n are here taken, not as constants over certain intervals of v (cf. Chap. VI), but rather as smooth slow-varying functions of v. But they vary so slowly, as compared with v, that they may be treated as constants in differentiating.

A $\delta\tau$ of 1° Fahrenheit would make:

$$\frac{\delta G}{G} = -(n-2)\frac{1}{2\tau_0}.$$

Substituting this value in equation 56:

$$\text{65.} \qquad \Delta X_\tau = -\int_{t_0}^{T} \frac{(n-2)}{2\tau_0} \mu' dt_\Delta.$$

Instead of using the tabular value of τ_0 in computing ΔX_τ, the present practice is to use the ground value (518.4°); and then treat ΔX_τ as though the value aloft had been used. This produces a slight error, which is probably less than the errors in our knowledge of the exact effects of temperature changes.

Formula 64 is simpler to compute and (as used) is more precise than 65, and a percentage change is simpler to use with a ballistic temperature; hence formula 64 will generally be preferred.

EFFECT OF NONSTANDARD DENSITY.

The weight of 1 cubic meter of air under standard conditions (59° Fahrenheit, 750 mm of mercury pressure, and 78 per cent saturation) is 1,203.4 gm. The values 59° and 750 mm were chosen by the Technical Staff for the reason that these values conform as nearly as feasible to the general practice of American, British, French, and Italian ballisticians during the World War. Seventy-eight per cent was chosen to conform with the present practice of the Coast Artillery. Ingalls's Table III also uses 59°, but uses 752 mm instead of 750 mm, thus giving a density of 1,206 gm per cubic meter. Entering this table with 59° and 750 mm (i.e., 29.53″), one gets 1.002, which for practical purposes is indistinguishable from unity.

The exponential law ($H = e^{-hy}$) was derived on the assumption that E varies as the density. And, of course, density varies as weight per any unit. Therefore H, E, density and the weight of a cubic meter at altitude y are proportional. Whence:

$$\therefore \frac{\delta H}{H} = \frac{\delta(\text{weight of a cubic meter})}{1203.4\ H}.$$

At the ground, H is unity. For an increase of 100 gm in the weight of a cubic meter of air at the ground:

$$\frac{\delta H}{H} = \frac{100}{1203.4} = 0.0831.$$

Substituting in equation 56:

66. $$\Delta X_H = 0.0831 \int_{t_0}^{T} \mu' dt_\Delta = 0.0831 \ (\mu_T - \mu_{t_0}).$$

Changes in atmospheric density aloft are converted into equivalent changes in atmospheric density at the ground, by the formula:

67. $$\frac{\text{Change aloft}}{H} = \text{equivalent ground change.}$$

For an increase of 1 per cent in atmospheric density.

$$\frac{\delta H}{H} = 0.01.$$

Substituting in equation 56:

68. $$\Delta X_H = 0.01 \int_{t_0}^{T} \mu' dt_\Delta = 0.01 \ (\mu_T - \mu_{t_0}).$$

It should be noted that this is the same formula as that for ΔX_C (formula 57), except that it is based on 1 per cent instead of 10 per cent and is opposite in sign.

Both the formula for 100 gm increase and the formula for 1 per cent increase will be used in the computations of Chapter XVI.

EFFECT OF A REAR WIND.[2]

To determine the effect of a wind moving in the same direction that the projectile is moving it is convenient to compare the motion of the projectile relative to the air and its motion relative to the earth. In other words, it is convenient to regard the motion of the projectile as though the projectile were moving along a trajectory relative to the air and as though this trajectory were itself moving as an entirety at the velocity of the wind.

If a 1 m/s wind is blowing in the direction of the x axis from time t_0 to time T, the x component of the velocity of the trajectory relative to the air at time t_0 and thereafter will be 1 m/s less than the x component of the velocity of the trajectory relative to the earth at that instant, which amounts to considering the x' of the projectile as receiving at time t_0 an increment ($\delta x'$) of -1; but the trajectory relative to the air is itself moving as an entirety at a rate of 1 m/s from time t_0 to time T. Thus the effects of a 1 m/s wind are two: (a) A minus 1 m/s $\delta x'$ at time t_0; (b) a plus horizontal movement of the entire system at a rate of 1 m/s for $T - t_0$ seconds. From formula 51, we take the terms:

$$\delta X = \int_{t_0}^{T} d\delta x + \nu \delta x'$$

[2] Compare "Physical Bases" (Ordnance Textbook 972), pp. 11–12.

Now as, by (b) above, δx is changing at the same rate as $t_\Delta, d\delta x = dt_\Delta$ By (a) above, $\delta x'_{t_0} = -1$. Therefore:

$$69. \qquad \Delta X_{wx} = \int_{t_0}^{T} dt_\Delta + \nu_{t_0}(-1) = T - t_0 - \nu_{t_0}$$

For a 1 m/s wind blowing throughout the trajectory:

$$70. \qquad \Delta X_{wx} = T - \nu_0$$

EFFECT OF A VERTICAL WIND.

Similar reasoning applies to the effects of a 1 m/s vertical wind. (1) A $\delta y'$ of minus 1 m/s at time t_0; (2) an upward movement of the entire system at a rate of 1 m/s for $T - t_0$ seconds. There are no variations in x or x'.

The effect of the decrease in y'_{t_0} is:

$$\delta X = \rho \ \delta y'_{t_0} = -\rho_{t_0}$$

The effect of the upward movement of the system is:

$$\delta X = \int_{t_0}^{T} h\mu \ d\delta y.$$

But, as δy is changing at the same rate as t_Δ, $d\delta y = dt_\Delta$; hence:

$$\delta X = h \int_{t_0}^{T} \mu dt_\Delta.$$

Thus the total range effect of a 1 m/s vertical wind, blowing from time t_0 to time T, is:

$$71. \qquad \Delta X_{wy} = h \int_{t_0}^{T} \mu dt_\Delta - \rho_{t_0}.$$

For a 1 m/s wind blowing throughout the trajectory:

$$72. \qquad \Delta X_{wy} = h \int_{0}^{T} \mu dt_\Delta - \rho_0.$$

CURVATURE OF THE EARTH.

In the Ingalls-Siacci system of ballistics, the practice was to compute a trajectory by means of rectangular-coordinate equations of motion based upon the assumption of a flat earth, and then apply the trajectory to the curved earth by referring it to a rectangular grid, whose x axis was tangent to the earth at the gun. Of course, in such a system, the farther one got from the gun, the farther the surface of the earth dropped away from the x axis; thus curvature of the earth increased the range by $\Delta X_k = 0.07848 \ R^2 \cot \omega$, where R was the radius of the earth.

But the present convention is that the trajectory, computed exactly as before, is now applied to the curved earth by referring it to the orthogonal curvilinear grid discussed in Chapter VII.

Under the tangent system, curvature was taken out of the range-firing range to get the tangent range, all range-table computations were based on the tangent range, and then the curvature correction had to be added in again.

Under the curved system, the curvature correction does not have to be either taken out or put in. Inasmuch as the *system* is curved, curvature of the *earth* may be disregarded.

SITE AND COMPLEMENTARY SITE.

In the Ingalls-Siacci system of ballistics, the practice was to treat a difference in altitude between gun and target, by assuming that a trajectory whose chord inclined from the gun to the target would have the same shape as though its chord were horizontal. This assumption was called the "theory of rigidity of the trajectory." It was defined by Col. Ingalls as assuming "that the relations existing between the elements of the trajectory and the chord representing the range are sensibly the same, whether the latter be horizontal or inclined to the horizontal."

It is obvious that, all other things being equal, a projectile will go farther if the ground slopes down from the gun, and less far if the ground slopes up from the gun, than if the ground were level. Assuming rigidity, and an angle of site (i. e., the inclination of a line drawn from the gun to the target; positive if up) of ϵ degrees, the correction to target range would exactly equal the error caused by a change of ϵ mils in ϕ.

$$\Delta X_{\epsilon} = [\Delta X_{\phi}]_{\delta\phi = \epsilon}$$

But the theory of rigidity is not sufficiently precise for some phases of modern fire (see Gunnery for Heavy Artillery, pp. 66–69; Gunnery for Field Service, pp. 23–25). The error in the theory of rigidity was formerly corrected for by expressing the effect of this error in terms of the change in elevation necessary to overcome it. This correction was known as the "complementary angle of site," the words "complementary angle" obviously not being used in their geometrical sense. The corrections for curvature, site, and complementary site were often combined into one correction.

The present practice is to derive a single range effect from the coordinates of a computed trajectory. The following formulas are used:

$$\tan \epsilon = \frac{y}{x}$$

73. $$\Delta X = x - X$$

The procedure is to plot ϵ against $x-X$ for each of the last few computed points of the trajectory, draw a smooth curve, and from it build up a site-correction table.

Whether the correction based on these formulas be called "the correction for site" or "the correction for site and complementary site," it covers *all* of the effects of a difference in altitude of gun and target.

PROBLEMS.

(55) Prove that—

$$\delta G = -v^2 \frac{dB}{dv} \frac{\delta s}{s_0}$$

See the foregoing derivation of formula 59 for effect of nonstandard temperature. Suggestion: by performing the indicated division, expand $\left(\frac{v}{s_0+\Delta s}\right)$ into two terms, one of which shall be $\frac{v}{s_0}$. By Taylor's Theorem for a single variable, expand $B\left(\frac{v}{s_0+\Delta s}\right)$ into a series of $B\left(\frac{v}{s_0}\right)$ and its derivatives.

(56) A shell weighing 15.96 pounds attains a range of 5,000 meters. A 10 per cent increase in ballistic coefficient would increase that range by 164 meters. A 1 m/s increase in muzzle velocity would increase the range by 2.6 meters. What range would be expected of by a projectile weighing 15.5 pounds, the powder charge and angle of departure remaining the same?

(57) A shell weighing 16.1 pounds is fired at the same elevation and in the same gun as that of the preceding problem. The muzzle velocity is measured and found to be 20 m/s above standard. What range should be expected?

QUESTIONS ON CHAPTER X.

1. What is the difference between ΔX and δX?
2. Give a complete list of the range effects discussed in this chapter, and state as to each the unit change on which it is based.
3. How does the correction for weight of projectile in range-table firings differ from the correction used in service firings, and why?
4. Between what limits should the formulas of this chapter be integrated?
5. What are the proper values of x', y', E, G, etc., to insert in the formulas of this chapter?
6. How is curvature of the earth corrected for?
7. Is complementary site included in the correction for site?
8. Given that $\Delta X_{wy} = 50$, map range$=10{,}000$, what range would you set for a vertical wind of $+2$ meters per second?
9. Would it be possible for ΔX_e to equal -25? Give reasons.

CHAPTER XI.

ANGLE OF DEPARTURE CORRECTION FORMULAS.

EFFECT OF JUMP.

Vertical jump equals angle of departure minus angle of elevation. It is therefore positive if up. The gun is laid by the angle of elevation, but it is the angle of departure which controls the shape of the trajectory.

Jump obviously affects the range through the formula for ΔX_{ϕ}. The object here will be, however, not to evolve a formula for the range effect of jump, but rather a formula for measuring the jump

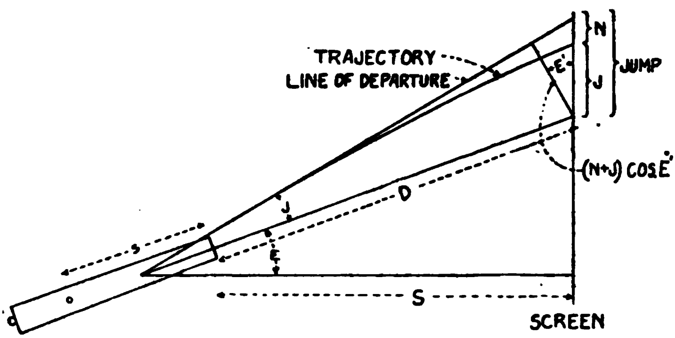


FIG 6

itself; inasmuch as it is $\bar{\phi}$, rather than X, which is always corrected for jump.

A special blank form is used at Aberdeen Proving Ground for the computation of jump. Most of the symbols used conflict with other symbols of ballistics, but are nevertheless here given just as they appear on the form.

S is the horizontal distance, in feet, from the muzzle to the jump screen.

E is the elevation at which the gun is laid for the jump firing.

$D = S \sec E$.

J is the measured vertical error, in inches, at the jump screen, + if up.

s is the distance from the trunnion axes to the muzzle.

t is the time of flight from muzzle to screen, i.e., approximately $\frac{D}{V}$.

N is the fall, in inches, due to gravity $= \frac{1}{2} gt^2 = \frac{1}{2}(32.16 \cdot 12 \cdot t^2) = 193\ t^2$.

The pivot of jump has been empirically determined to be about halfway from the trunnions to the muzzle, rather than at the trunnions, jump being due to a combination of rotation about the trunnions and other motions, principally recoil.

j is the angle of jump.

$$\sin j = \frac{(N+J)\cos(E+J)}{12\left(D+\frac{s}{2}\right)}$$

the reason for the 12 being to reduce $\left(D+\frac{s}{2}\right)$ to inches like $(N+J)$.

But j is so small that $\cos(E+j)$ is practically $\cos E$, and j (in minutes) is practically $\frac{\sin j}{\sin 1'}$. Sin $1'$ is $\frac{1}{3438}$. Therefore

74. $$j = \frac{3438\,(N+J)\cos E}{12\left(D+\frac{s}{2}\right)} = \frac{573\,(N+J)}{(2D+s)\sec E}.$$

EFFECT OF CANT.

Although the most marked effect of cant (i. e., trunnions out of level) is the effect on deflection, yet it also has a slight effect on the angle of departure.

See the "effect of cant" in Chapter XII. From the development there, it is evident that cant has the following effect on angle of departure:

$$\Delta\phi \text{ (in mils)} = -1019 \cos i\,(\tan\phi - \tan E)$$

Converting into minutes,

75. $$\Delta\phi = -3438 \cos i\,(\tan\phi - \tan E)$$

The i of these formulas is the angle of cant, and has no relation to the form factor i, nor to the i of the jump formula.

When laying a gun with a quadrant or with a sight shank that is capable of being leveled, the *indicated* elevation is the *true* elevation, and no range correction for cant need be made. Even in the case of guns laid by range drum or by nonlevelable scale, the correction is apt to be inappreciable unless the trunnions or base ring are badly out of level.

ANGLE OF SITE.

The point of splash in range firing is of course at a lower level than the gun. The exact difference may be ascertained by comparing the height of the trunnions above mean low water and the height of the tide.

The difference will be so slight that the "theory of rigidity" (see p. 68, ante) may be assumed without necessitating any "complementary" correction.

This difference, divided by the range (being careful to convert to the same units), gives the tangent of the angle of site. (See the definition of "angle of site" in Chapter V.) As small angles are approximately proportional to their tangents, and as the tangent of 1′ is $\frac{1}{3438}$, then the angle of site (in minutes) is equal to the above ratio multiplied by 3438.

This angle must be *added to* the quadrant angle of departure to give the ϕ which would have produced the same X, if firing on the level.

IN GENERAL.

It is often convenient to know what elevation correction is necessary to offset any one of a number of nonstandard conditions affecting range.

Suppose, for instance, that the disturbing cause is a 1 m/s helping wind. This is found, by formula 70, to produce a certain ΔX. Find, by formula 53, what $\Delta\phi$ will produce ΔX, numerically equal but opposite in sign. This $\Delta\phi$ is the desired correction.

Similarly can be found the elevation correction necessary to offset any other disturbing nonstandard condition.

PROBLEM.

(58) A gun is laid at an elevation of 2°. From the muzzle to the jump screen the horizontal distance is 100 feet. The length of the gun from trunnion axis to muzzle is 20 feet. The shell pierces the jump screen 30 inches above the point of bore sight. The muzzle velocity is 2,000 foot-seconds. What is the jump?

QUESTIONS ON CHAPTER XI.

1. Why is the pivot of jump considered to be halfway between the trunnions and the muzzle?
2. Define vertical jump.
3. Define cant.
4. Define angle of site.
5. How do you find the elevation correction necessary to offset the range effect of some disturbing cause?
6. In deriving the jump formulas, why is g taken as 32.16, instead of 9.80, as in the rest of the book?

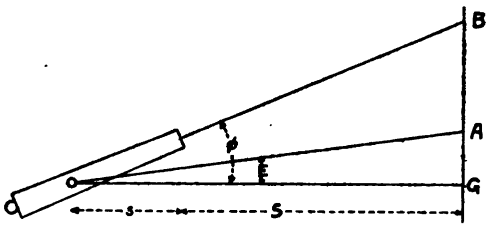
CHAPTER XII.

DEFLECTION FORMULAS.

EFFECT OF CANT.

Cant occurs when one trunnion is higher than the other. The tangent of i, the angle of cant, is obtained by dividing "right wheel above left" by the "distance between levels."

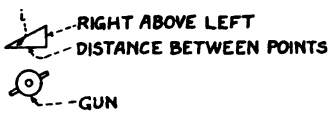
Suppose a gun is elevated E degrees and bore-sighted at a point on a jump-screen $S+s$ feet from its trunnions, and then is slowly elevated to ϕ degrees. (See Fig. 7.) If the trunnions are level, the line of bore-sight will travel vertically up the screen a distance equal to



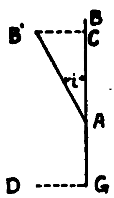
SIDE VIEW OF GUN AND SCREEN (TRUNNIONS LEVEL)
FIG. 7

$$(S+s)\ (\tan\phi-\tan E).$$

Notice that in the case of cant the pivot is at the trunnions. But this

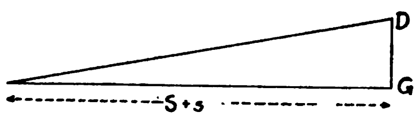


REAR VIEW OF GUN
(TRUNNIONS INCLINED)
FIG 8



REAR VIEW OF SCREEN
FIG 9

vertical line will be tilted i degrees to the left, if the trunnions are tilted through i degrees by raising the right trunnion above the left. (See Figs. 8, 9, 10.) The line of bore-sight will now cut the screen $(S+s)$ $(\tan\phi-\tan E)\sin i$ feet to the left of its original position, and hence will have a deflection



GROUND PLAN
FIG 10

whose sine equals

$$(S+s)\ \frac{(\tan\phi-\tan E)}{(S+s)}\ \sin i,\ \text{ i. e.,}$$

$$\sin D=-(\tan\phi-\tan E)\sin i,$$

the minus sign being introduced because, deflection to the right being taken as positive, and "right wheel above left" being taken as positive for cant, a positive cant produces a negative deflection.

Now, since small angles vary approximately as their sines, and since the sine of 1 mil is $\frac{1}{1019}$,

$$D \text{ (in mils)} = \frac{\sin D}{\sin 1 \text{ mil}} = 1019 \sin D$$

76. $$D = -1019 (\tan \phi - \tan E) \sin i.$$

EFFECT OF LATERAL JUMP.

Let I be the lateral jump, in inches, measured on the jump screen, plus if to the right. Let i be the deflection angle of side jump. Do not confuse this i with the form factor i, nor with the i of the cant formula. The other symbols are the same as those used in the discussion of jump in Chapter XI. Notice that here, as in the case of vertical jump, the pivot is taken as half way between the trunnions and the muzzle.

The horizontal distance, in feet, from the pivot to the screen is $S+\frac{s}{2} \sec E$. But $\frac{s}{2}$ is so small compared with S, and sec E is so nearly unity, that the expresssion may be written $S+\frac{s}{2}$ without material error.

$$\sin i = \frac{I}{12\left(S+\frac{s}{2}\right)}$$

$$i \text{ (in mins.)} = \frac{\sin i}{\sin 1'} = \frac{3438\, I}{12\left(s+\frac{s}{2}\right)} = \frac{573\, I}{2\, S+s}.$$

77. $$i \text{ (in mils)} = \frac{170\, I}{2\, S+s}.$$

Lateral jump is usually not computed, but is included in the "drift" as tabulated in range tables.

EFFECT OF CROSS WIND.

A 1 m/s cross wind is treated much the same as a 1 m/s range wind or a 1 m/s vertical wind (see Chap. X) A cross wind is positive, if blowing from left to right.

A+1 m/s cross wind, blowing from time t_0 to time T, causes the trajectory relative to the air to bend abruptly to the left at time t_0, by an angle whose tangent is $\frac{1}{x'}$. Therefore the deflection in meters at time T due to this cause alone will be to $x_T - x_{t_0}$, as 1 to x'_{t_0}; i. e., $\frac{(x_T - x_t)}{x'_{t_0}}$ to the left.

But the trajectory relative to the air will itself be moving to the *right* at a rate of 1 m/s for $(T-t_0)$ seconds, or a distance of $(T-t_0)$ meters. Therefore the net deflection, in meters, is:

78. $$D_w = T - t_0 - \frac{X-x}{x'}$$

the values of x and x' being those at time t_0.

In mils, assuming the wind to blow throughout the trajectory,

79. $$D_w = 1{,}019\left(\frac{T}{X} - \frac{1}{x'_0}\right).$$

DRIFT.

Drift is due to the gyroscopic action of the projectile.

Drift is at present treated as wholly empirical. In range firing the total angular deviation from the line of fire is measured (positive if to the right, negative if to the left). From this is subtracted algebraically the effects of cant and cross wind. The remainder is the drift (plus lateral jump, of course, which, for range-table purposes, is usually included in the tabulated value of drift).

PROBLEM.

(59) In the firing of problem 58, the right trunnion was 1 inch above the left, the distance between the points where the levels were taken being 3 feet. The shell pierced the jump-screen 9 inches to the left of the point of bore-sight. The gun is then elevated to 10° and attains a range of 6,421 meters. There is a 9.5 m/s cross wind blowing from left to right. The time of flight is 17.1 seconds. The shell lands 40 meters right. What is the drift in mils? For the purposes of this problem do not include the lateral jump in the drift.

QUESTIONS ON CHAPTER XII.

1. If deflection is given in meters, how should it be converted to mils?
2. Why, in the case of cant, is the pivot considered to be at the trunnions?
3. What is the cause of drift?
4. How is drift determined?
5. What does the range-table value of drift include, besides drift proper?

CHAPTER XIII.

ROTATION OF THE EARTH.[1]

IN GENERAL.

The rotation of the earth has two effects on a projectile in flight: (*a*) The higher the projectile goes, the more must its velocity be altered, in order to maintain the same linear velocity relative to the

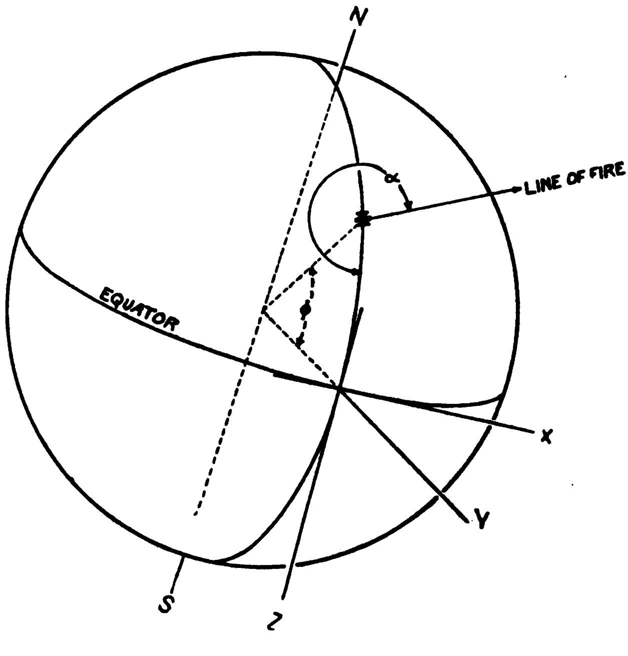


FIG 11

earth; (*b*) centrifugal force offsets to some extent the attraction of gravity.

One of the assumptions on which the computation of a standard trajectory is based is (see Chap. V): "2. The earth is motionless. (The average effect of the rotation of the earth on gravity is included in the assumed value of g.)"

The object of the present chapter will be to ascertain the difference between equations of motion based on the assumption of a motionless earth and those based on the assumption of a rotating earth, to separate out from this difference such effects as the standard

[1] For a nonmathematical treatment of this subject from another viewpoint, see "Physical bases" (Ordnance Textbook 972), pp. 9–11.

trajectory includes in g_0, and to base a rotation correction on the remainder.

Inasmuch as most discussions of the rotation of the earth adopt the convention of rectangular axes (the x axis being tangent to the earth at the gun), and as the corrections thus derived can be shown to be equally applicable to the convention of a curved grid (with x lines concentric and y lines radial), this chapter will develop the corrections on the rectangular basis.

Let us consider a projectile fired from a point whose latitude is l, in a direction whose azimuth is α, measured clockwise from the south.

Let us consider two sets of rectangular axes, coincident at the instant the gun is fired. The origin is on the equator at the same longitude and altitude above sea level as that of the gun; the y axis is vertical upward; the x axis is horizontal to the east; the z axis is horizontal to the south. Inasmuch as the orbital motion of the earth has a negligible effect upon the trajectory, we shall conceive of the axis of rotation of the earth as fixed in space.

Now consider that one set of axes (designated by m) moves with the earth; and that the other set (designated by f), although coincident with the moving set when the projectile leaves the gun, thereafter remains fixed in space, and does *not* rotate with the earth.

Let t be the time which has elapsed since the gun was fired. Let R be the radius of the earth. Do not confuse this R with the symbol for retardation. Let Ω be the angular velocity of rotation of the earth.[2]

At time t_Δ, the angle between the two sets of axes is Ωt_Δ; so, by the familiar formulas of analytic geometry:

80.
$$\begin{cases} x_m = x_f \cos \Omega t_\Delta - (y_f + R) \sin \Omega t_\Delta \\ y_m + R = x_f \sin \Omega t_\Delta + (y_f + R) \cos \Omega t_\Delta \\ z_m = z_f \end{cases}$$

The subscript m means referred to the moving set of axes. The subscript f means referred to the fixed set of axes.

Differentiating equations 80, we get:

81.
$$\begin{cases} x'_{mm} = x'_{ff} \cos \Omega t_\Delta - y'_{ff} \sin \Omega t_\Delta - y_m \Omega - R\Omega \\ y'_{mm} = x'_{ff} \sin \Omega t_\Delta + y'_{ff} \cos \Omega t_\Delta + x_m \Omega \\ z'_{mm} = z_{ff} \end{cases}$$

The meaning of the double subscript is as follows: For instance, x'_{ff} means the time derivative of x_f, which derivative is left referred to the fixed system of axes. If we were to transform x'_{ff}, by formulas analogous to equations 80, so as to refer to the moving system, we should get x'_{fm}.

---

[2] Since there are 86,164 mean solar seconds in a sidereal day,

$$\Omega = \frac{2\pi}{86,164} = .00007292 \text{ radians per second.}$$

Differentiating equation 81, we get:

82. $$\begin{cases} x''_{mmm} = x''_{fff} \cos \Omega t_{\Delta} - y''_{fff} \sin \Omega t_{\Delta} - 2y'_{mm}\Omega + x_m\Omega^2 \\ y''_{mmm} = x''_{fff} \sin \Omega t_{\Delta} + y''_{fff} \cos \Omega t_{\Delta} + 2x'_{mm}\Omega + (y_m + R)\Omega^2 \\ z''_{mmm} = z''_{fff} \end{cases}$$

The meaning of the triple subscript is as follows: For instance, x''_{fff} means the time derivative of x'_{ff}, which derivative is left referred to the fixed system of axes. If we now transform x''_{fff} by formulas analogous to equations 80, so as to refer to the moving system of axes, we shall get x''_{ffm}. These formulas are:

83. $$\begin{cases} x''_{ffm} = x''_{fff} \cos \Omega t_{\Delta} - y''_{fff} \sin \Omega t_{\Delta} \\ y''_{ffm} = x''_{fff} \sin \Omega t_{\Delta} + y''_{fff} \cos \Omega t_{\Delta} \\ z''_{ffm} = z''_{fff} \end{cases}$$

Substituting from equations 83 in equation 82, we get:

84. $$\begin{cases} x''_{mmm} = x''_{ffm} - 2y'_{mm}\Omega + x_m\Omega^2 \\ y''_{mmm} = y''_{ffm} + 2x'_{mm}\Omega + y_m\Omega^2 + R\Omega^2 \\ z''_{mmm} = z''_{ffm} \end{cases}$$

The symbols with the subscripts *mmm* are the components of relative acceleration; those with the subscripts *ffm* are the components of absolute acceleration; both being referred to the moving set of axes, which set is that used in computing trajectories.

Equations 84 can now be simplified by dropping the negligible terms $x_m\Omega^2$ and $y_m\Omega^2$. Even with a gun firing 100 miles, neither of these terms could exceed 0.00001 on any part of the trajectory. Thus:

85. $$\begin{cases} x''_{rel} = x''_{abs} - 2y'_{rel}\Omega \\ y''_{rel} = y''_{abs} + 2x'_{rel}\Omega + R\Omega^2 \\ z''_{rel} = z''_{abs} \end{cases}$$

Now, regardless of whether we consider the earth as moving or as motionless, what we are interested in is the motion of the projectile relative to the earth. If we assume that the earth is motionless, all terms involving Ω in equation 85 would drop out, and we should have:

86. $$\begin{cases} x''_{rel} = x''_{abs} \\ y''_{rel} = y''_{abs} \\ z''_{rel} = z''_{abs} \end{cases}$$

If we consider the earth motionless, except for its centripetal acceleration, the result is to add $R\Omega^2$ to the right side of the second equation of 86. Thus:

87. $$\begin{cases} x''_{rel} = x''_{abs} \\ y''_{rel} = y''_{abs} + R\Omega^2 \\ z''_{rel} = z''_{abs} \end{cases}$$

Equations 87 represent relative accelerations under the standard assumptions of trajectory computation. Equations 84 represent relative accelerations as they actually exist. Therefore the effect of the rotation of the earth may be found by subtracting equations 87 from equations 85.

88.
$$\begin{cases} \delta x'' = -2y'\Omega \\ \delta y'' = +2x'\Omega \\ \delta z'' = 0 \end{cases}$$

Let us now shift the axes until they become the axes used in trajectory computation. First rotate the system northward about the center of the earth through an angle of l degrees. The origin will then coincide with the gun. Equations 88 become:

89.
$$\begin{cases} \delta x'' = -2\Omega(y' \cos l + z' \sin l) \\ \delta y'' = +2\Omega(x' \cos l) \\ \delta z'' = +2\Omega(x' \sin l) \end{cases}$$

Let us now rotate the x and z axes clockwise around the y axis, through an angle of $(\alpha - 270)$ degrees. The x axis will now point in the direction of the line of fire; z' will equal zero. Equations 89 therefore become:

90.
$$\begin{cases} \delta x'' = +2\Omega y' \cos l \sin \alpha \\ \delta y'' = -2\Omega x' \cos l \sin \alpha \\ \delta z'' = +2\Omega(y' \cos l \cos \alpha + x' \sin l) \end{cases}$$

Now equations 90 were derived on the assumption of a rectangular system of coordinates, whose x axis is tangent to the earth at the gun (the "tangent method") instead of on the assumption of a system whose abscissas are measured along a circle concentric with the earth and whose ordinates are measured radially from that circle (the "curved method").

It now remains to be shown that equations 90 apply equally to the curved method. For this purpose equations 90 must be derived so as to relate to instantaneous axes respectively vertical and horizontal at the point where the projectile is at that instant,[3] instead of axes respectively vertical and horizontal at the gun.

In other words, any arbitrary point on the trajectory will be taken, and equations 90 will be derived with respect to motion at that point. Then, as that point was *any* point on the trajectory, equations 90 will apply to *all* points on the trajectory. That point will be called the instantaneous point.

Now revert to page 77, and consider (in place of the two sets of axes there mentioned) two sets coincident when the projectile reaches

[3] That is, the same axes as those of equations 111 in Supplement E.

the instantaneous point. The origin is on the Equator at the same longitude and same altitude above sea level as the instantaneous point. Except as otherwise stated, the development is the same as the preceding.

t_Δ is the time which has elapsed since the projectile was at the instantaneous point. R is the distance from the center of the earth to the instantaneous point.

Equations 80 and 84 are developed for a point on the trajectory very near to the instantaneous point. This near point is then allowed to approach the instantaneous point as a limit, with the result that the terms $x_m\Omega^2$ and $y_m\Omega^2$ become zero and equations 84 become equations 85.

Equations 86 and 90 evolve as before,[4] with no change in their derivation, except that $R\Omega^2$ now represents the centripetal acceleration at the instantaneous point instead of at the surface of the earth. But as gravity is assumed to be constant, regardless of altitude (which assumption can be shown to cause an inappreciable error even with the most powerful guns), $R\Omega^2$ can still be considered to be included in g.

Equations 90 are thus seen to represent the increments of acceleration due to rotation, and to be correct for either the tangent or the curved method. In the tangent method they refer to a rectangular coordinate system whose y axis is vertical at the gun. In the curved method they refer to the directions which are horizontal and vertical at the particular point on the trajectory.

RANGE EFFECT.

The range effect of rotation of the earth may be found by inserting the values of $\delta x''$ and $\delta y''$ from equations 90 into equation 54 of Chapter X, with the following result:

91. $$\Delta X_\Omega = 2\Omega \cos l \sin \alpha \int_0^T (\nu y' - \rho x') dt_\Delta$$

2Ω equals 0.00014584.

DEFLECTION EFFECT.

Consider now the deflection effect (D in meters) of an increment in lateral velocity ($\delta z'$) occurring at time t_Δ. Then:

$$\frac{D}{X-x} = \frac{\delta z'}{x'}$$

Whence:

$$D = \frac{X-x}{x'} \delta z'.$$

[4] In the curved method, equations 90 relate to instantaneous cartesian axes which are horizontal and vertical at the instantaneous location of the projectile. To transform them, so as to relate to the curvilinear grid of the curved method, we might use equations 120 of supplement E. But if the increments be regarded as analogous to infinitesimals of the first order, the only effect of such a transformation will be to add infinitesimals of a higher order, containing $\frac{1}{R}$, which terms can hence be disregarded. Hence equations 90 may be considered as remaining unchanged.

By differentiation, followed by integration, as in the derivation of equation 21, we get the following expression for the deflection effect of an increment in lateral acceleration, occurring throughout the trajectory:

92. $$D=\int_0^T \frac{X-x}{x'}\,\delta z''\,dt_\Delta.$$

Whence, substituting the value of $\delta z''$ from equations 90:

93. $$D_\Omega=2\Omega\cos l\cos\alpha\int_0^T \frac{X-x}{x'}\,y'dt_\Delta+2\Omega\sin l\int_0^T (X-x)\,dt_\Delta.$$

For convenience, the following "rotation coefficients" have been adopted:

94. $$\begin{cases} A=0.00014584\int_0^T (x'\rho-y'\nu)\,dt_\Delta. \\ B=0.00014584\int_0^T (X-x)\,dt_\Delta. \\ C=0.00014584\int_0^T \frac{X-x}{x'}\,y'dt_\Delta. \end{cases}$$

Note that B is not the B function of atmospheric resistance, and that C is not the ballistic coefficient.

Therefore the range and deflection effects of rotation of the earth become, from equations 91 and 93:

95. $$\begin{cases} \Delta X_\Omega=A\cos l\cos\alpha. \\ D_\Omega=B\sin l+C\cos l\cos\alpha. \end{cases}$$

A close approximation to the three rotation coefficients is the value which would obtain in a vacuum,[5] namely:

$$A=\Omega XT\left(\cot\omega-\frac{1}{3}\tan\varphi\right).$$

$$B=\Omega XT.$$

$$C=\frac{1}{3}\,\Omega XT\tan\varphi.$$

PROBLEMS.

(60) Prove that $x_m\Omega^2<0.00001$, as stated on page 78.
(61) Derive equations 89 from equations 88.
(62) Derive equations 90 from equations 89.

[5] See "Physical Bases" (Ordnance Textbook 972), p. 11.

Questions on Chapter XIII.

1. What are the two effects of rotation of the earth?
2. For which of these need the range be corrected?
3. Do the formulas derived in this chapter apply to the "tangent method" or to the "curved method" of trajectory computation?
4. What acceleration is used in the computation of a standard trajectory?
5. What acceleration should be used to account for all the effects of rotation of the earth?
6. Explain the ratio given under "Deflection Effect."
7. Explain the figure 0.00014584 in equations 94.
8. Firing in vacuo, what value of φ would cause the range effect of rotation of the earth to be zero?

CHAPTER XIV.

COMPUTATION OF DIFFERENTIAL CORRECTIONS.

The computation of differential corrections is based upon the following equations, derived in Chapter IX.

$$
96. \quad \begin{cases}
\mu'' = 2hEy'\mu + \left[E - hy' - gy'\left(\dfrac{d\log G}{v\,dv}\right)\right]\mu' + 2\,Ex' \\
\mu' = \displaystyle\int^{t_0} \mu'' dt_\Delta \\
\mu = \displaystyle\int^{t_0} \mu' dt_\Delta \\
\rho = \dfrac{1}{g}\,(x' + hy'\mu + \mu') \\
\nu = -\dfrac{y'}{x'}\rho - \dfrac{\mu'}{Ex'}
\end{cases}
$$

The method of procedure is to integrate μ'' ahead to get an approximate value of μ', integrate the approximate μ' to get an approximate value μ, and then get an approximate value of μ'' by substituting the approximate values of μ and μ'' in the first above formula. Then integrate μ'' to get μ', and μ' to get μ; continuing the process of successive approximations for any one line until the values check. Thus we employ numerical integration and successive approximations in a manner very similar to the method of computing a standard trajectory.

The data is taken from a trajectory computed by the methods of Chapter VIII. The computing form is made by taking a sheet of paper about the size of the trajectory sheet of the trajectory computations, having horizontal blue lines one-fourth inch apart. Rule a vertical line 1½ inches from the left-hand margin, then one one-half inch from the first, then one one-half inch from that, then every inch or three-fourths inch across the page.

Two special tables are used, being entered with $\dfrac{v^2}{100}$ as an argument. These tables are:

Table of $f\,(v) = 10^6\left(h + g\dfrac{G'}{vG}\right)$

Table of $(n-2)$

On the computing sheet, the times run *across* the page instead of *down* it, and run from T to 0, instead of from 0 to T.

The first four columns are filled in as follows:

	No.	Next.	Number of decimals.	T		1	0
t_0	1	4	1	....	....	....	....
μ (approx.)	2	13	2	....	....	....	....
μ' (approx.)	3	2	2	....	....	....	....
x'	4	....	2	....	....	....	....
y'	5	....	2	....	....	....	....
Ex'	6	....	2	....	....	....	....
Ey'	7	42	2	....	....	....	....
$10^6.\ 2h\ Ey'$	8	10	1	....	....	....	....
$10^6\left(h+g\ \frac{d\log G}{v\ dv}\right)$	9	8	1	....	....	....	....
y' ( )	10	12	4	....	....	....	....
E	11	31	4	....	....	....	....
$(1)=E-y'$ ( )	12	3	4	....	....	....	....
$2h\ Ey'\ \mu$	13	....	2	....	....	....	....
$+\ (1)\ \mu'$	14	16	2	....	....	....	....
$+2\ Ex'$	15	11	2	....	....	....	....
$\mu''=$sum	16	....	2	....	....	....	....
a	17	....	2	....	....	....	....
b	18	....	2	....	....	....	....
$\mu'=\mu_T-\int_{t_0}^{T}\mu''\ dt._\Delta$	19	....	2	....	....	....	....
a	20	....	2	....	....	....	....
b	21	....	2	....	....	....	....
$\mu=\mu_T-\int_{t_1}^{T}\mu' dt_\Delta$	22	....	1	....	....	....	....
a	23	3, (24)	1	....	....	....	....
$(2)=0.01\ (\mu_T-\mu_{t0})$	24	....	2	....	....	....	....
Dens.$=(2)\times 8.310$	25	27	1	....	....	....	....
x'	26	15	1	....	....	....	....
$+hy'\mu$	27	....	1	....	....	....	....
$+\mu'$	28	....	1	....	....	....	....
sum	29	....	1	....	....	....	....
$\rho=\frac{\text{sum}}{g}$	30	32	2	....	....	....	....
$\frac{y'}{x'}$	31	60	4	....	....	....	....
$(3)=\rho\ \frac{y'}{x'}$	32	....	2	....	....	....	....

	No.	Next.	Number of decimals.	T		1	0
$(4)=\frac{\mu'}{Ex'}$	33	....	2	....	....	....	....
$\nu=-(3+4)$	34	....	2	....	....	....	....
$(5)=10^{-4}\ x'\rho$	35	....	2	....	....	....	....
$(6)=10^{-4}\ y'\nu$	36	....	2	....	....	....	....
$(7)=(6-5)$	37	....	2	....	....	....	....
a	38	....	2	....	....	....	....
b	39	....	2	....	....	....	....
$(8)=\int_{t_0}^{T}(7)\ dt_\Delta$	40	....	2	....	....	....	....
a	41	43	2	....	....	....	....
$(9)=0.0001\ (X-x)$	42	50	2	....	....	....	....
a	43	....	2	....	....	....	....
$(10)=\int_{t_0}^{T}(9)dt_\Delta$	44	....	2	....	....	....	....
a	45	....	2	....	....	....	....
$(11)=\frac{y'}{x'}\ (9)$	46	....	2	....	....	....	....
a	47	....	2	....	....	....	....
$(12)=\int_{t_0}^{T}(11)\ dt_\Delta$	48	....	2	....	....	....	....
a	49	51	2	....	....	....	....
$(13)=\frac{(X-x)}{x'}$	50	26	2	....	....	....	....
$T-t_0$	51	53	2	....	....	....	....
$W_z=T-t_0-(13)$	52	54	2	....	....	....	....
$W_x=T-t_0-\nu$	53	52	2	....	....	....	....
$(14)=0.001\ \mu$	54	....	2	....	....	....	....
a	55	....	2	....	....	....	....
$(15)=\int_{t_0}(14)\ dt_\Delta$	56	....	2	....	....	....	....
a	57	....	2	....	....	....	....
$(16)=1.036\ (15)$	58	....	2	....	....	....	....
$W_y=(16)-\rho$	59	61	2	....	....	....	....
n-2	60	9	3	....	....	....	....
$(17)=(n-2)\ \mu'$	61	....	2	....	....	....	....
a	62	....	2	....	....	....	....
b	63	....	2	....	....	....	....
$(18)=\int_{t_0}^{T}(17)\ dt_\Delta$	64	....	2	....	....	....	....
a	65	....	2	....	....	....	....
$\Delta X_\tau=0.005\ (18)$	66	Finals.	3	....	....	....	....

The fourth column above ("number of decimals") need not be copied. It is given above merely to indicate the correct number of decimals to which to carry the computations.

Put the total time of flight (T) at the head of the fourth column, and at the head of the succeeding columns put all preceding whole values of t, from T back to zero.

Fill in lines 4, 5, 6, 7, 42, 50, 26, 15, 11, and 31 with data from the trajectory computations, and lines 60 and 9 with data from the special tables. It is optional whether or not to fill in on the trajectory computation sheets themselves the values of E, $X-x$, $\frac{y'}{x'}$, etc.

Line 8 is computed from line 7, using $+0.0001036$ as the value of h. Line 10 is computed from lines 5 and 9. Line 12, from lines 10 and 11.

The numbers in the column labeled "Next" show which line to proceed to, after completing the line in which the number occurs. In the absence of any number in the "Next" column, proceed to the next line below. These numbers have no relation to the *italicized* numbers in parenthesis in the left-hand column.

The work, up to line 12, has been line by line across the page. But from now on (until line 23 is completed in all columns), we proceed to solve each column by numerical integration and successive approximations.

The procedure is to start with values for μ and μ', obtained from the following formulas. For time T, the following are precise:

$$\mu=-\frac{1}{h}\cdot\frac{x'_T}{y'_T}$$

$$\mu'=0$$

$$\mu''=0.$$

For the next two columns the following are approximate:

$$\mu''=2\,gE_T\,(T-t)\,\frac{x'_T}{y'_T}$$

$$\mu'=-\frac{\mu''}{2}\,(T-t)$$

$$\mu=\mu_T-\frac{\mu'}{3}\,(T-t).$$

The values of μ are useful merely for a check.

Thereafter the approximate values of μ and μ' (to be entered in lines 2 and 3) may be found by integrating μ'' ahead by formula 35:

$$\int_{t-i}^{t+i} f\,dt=2\,if_t+\frac{i}{3}\,(b_t+c_t+d_t\;.\;.\;.\;)$$

to get the increment of μ'; and then integrating μ' by formula 34:

$$\int_{t-i}^{t} f\,dt = i\left(f_t - \frac{1}{2}a_t - \frac{1}{12}b_t - \frac{1}{24}c_t - \frac{1}{40}d_t\right)$$

to get the increment of μ. μ is positive throughout; μ' is negative throughout.

Although we are integrating from time T toward time zero, the student should proceed exactly as in the method laid down in Chapter VIII. Wherever it is necessary to change signs in integrating, that fact is indicated by a minus before the $\int_{t_0}^{T}$ in the left-hand column in lines 19 and 22. The first, second, etc., differences will be formed, proceeding *from* T *toward* zero[1]; but the time interval (i) will be treated as positive. For a mathematical explanation of all this, see supplement B.

Note that the time interval between the first and second column is different from the uniform time interval of succeeding columns. The first difference obtained by differencing the first and second columns is used only in integrating for values for the second column. There should be no second difference in the third column, no third difference in the fourth column, etc.

The procedure, therefore, is as follows: Insert the precise values for time T in lines 2, 3, 19, and 22 of the T column. To get the first precise value of μ, it will be necessary to use straight multiplication and division, preferably on a calculating machine, as logarithms will not give results precise to enough places. Insert the approximate values in lines 2 and 3 of the next two time columns. Use these values in lines 13 and 14, and thus get three values of μ'' in line 16. Form the first differences of μ'', and integrate for the increment of μ', using formula 34 above. Similarly integrate for the increment of μ. The values thus obtained will usually check so closely with the approximate values, that μ'' will not have to be recalculated.

Start each succeeding column by integrating μ'' ahead to get the approximate increment of μ'. Add this algebraically to the last precise value of μ', to get the new approximate μ'. Integrate this approximate μ' to get the approximate μ. Use these values in lines 13, 14, and 16, and proceed as before.

After completing the fourth or fifth column of computations, it may be well, though not essential, to extrapolate back for values of μ'', μ', and their first, second, third, etc., differences for a time equal to the next whole number larger than T, i. e., one whole second larger than the time recorded at the head of the second column of computations.

[1] That is, a_4 equals $f_4 - f_5$. See the footnote on p. 30.

Starting from this point we can now perform our integrations, with a constant time interval and a full set of a, b, c, etc., for use in the integration. This results in smoothing out the initial steps and in giving them a greater precision.

When μ' and μ have been computed clear across the page, we are in a position to compute the differential corrections. All of the following work is done a line at a time clear across the page.

First comes the effect of an increase of 100 gm in the weight of a cubic meter of air (using lines 24 and 25); from formula 66:

$$\Delta X_H = 0.0831\ (\mu_T - \mu_t).$$

The value for time zero is the range-table value, the other values being used as a basis for weighting-factor curves, as will be explained later (see Chap. XV).

Then comes (in lines 26–34) the computation of the auxiliary variables ρ and ν.

Then come (in lines 35–49) the three rotation coefficients, formula 94:

$$\begin{cases} A = 0.00014584 \int_0^T (x'\rho - y'\nu)\, dt_\Delta \\ B = 0.00014584 \int_0^T (X - x)\, dt_\Delta \\ C = 0.00014584 \int_0^T \left(\frac{X-x}{x'}\right) y'\, dt_\Delta \end{cases}$$

which enter into the formulas for the range and deflection effects of the rotation of the earth. A, B, and C are respectively obtained by multiplying by 1.4584 the values, for time zero, in lines 40, 44, and 48.

Then come (in lines 50–59) the effect on deflection of a cross wind, and the effect on range of a range wind or a vertical wind, 1 m/s being taken as the unit wind; formulas 78, 69, and 71:

$$D_w = T - t_0 - \left[\frac{X-x}{x'}\right]_{t_0}$$

$$\Delta X_{wx} = T - t_0 - \nu_{t_0}$$

$$\Delta X_{wy} = -\rho t_0 + h \int_{t_0} \mu dt_\Delta$$

The value for time zero is the range-table value, the other values beng used as a basis for weighting-factor curves, as will be explained later. (See Chap. XV).

Then comes (lines 60–66) the range effect of the change in elasticity due to a 1 per cent increase in temperature; formula 64:

$$\Delta X_\tau = -0.005 \int_{t_0}^T (n-2)\ \mu' dt_\Delta$$

The value for time zero is the range-table value, the other values being used as a basis for weighting-factor curves, as will be explained later. (See Chap. XV.)

This completes the calculations of the columns. We now proceed to certain formulas which are based merely upon initial or terminal data; formulas 53, 52 and 57:

$$\Delta X_\phi = \frac{V}{1019}\left[\rho \cos\phi - \nu \sin\phi\right]_{t_0=0}$$

$$\Delta X_v = [\nu \cos\phi + \rho \sin\phi]_{t_0=0}$$

$$\Delta X_c = 0.1\ (\mu_0 - \mu_T)$$

These are, respectively, the effects of a 1-mil increase in ϕ, a 1 m/s increase in V, and a 10 per cent increase in C.

They can be checked by the following (see problems 54 and 53):

$$[\rho \cos\phi - \nu \sin\phi]_{t_0=0} = \left[\frac{\rho}{\cos\phi} + \frac{\mu' \sin\phi}{Ex'}\right]_{t_0=0}$$

$$\Delta X_v = -\left[\frac{\mu' \cos\phi}{Ex'}\right]_{t_0=0}$$

$\frac{\mu'}{Ex'}$, for time zero being taken from line 33.

The computation sheet is completed by the following summary of effects:

1 m/s change in V.
1 mil change in ϕ.
Rear wind, 1 m/s.
Vertical wind, 1 m/s.
Cross wind, 1 m/s.
100 gm/m^3 increase in density.
10 per cent increase in C.
1 per cent increase in absolute temperature.
Rotation A.
Rotation B.
Rotation C.

PROBLEM.

(63) Compute the differential corrections complete from the following data: $V=579.1$ m/s; $C=2.3$; $\phi=5°$; T$=8.824$ sec.

t	0	1	2	3	4
x'	576.9	520.2	473.0	433.7	401.2
y'	50.5	36.2	23.5	12.2	1.8
E	0.1075	0.0994	0.0910	0.0823	0.0732
$\frac{v^2}{100}$	3354	2719	2242	1882	1609
x	0	548	1044	1496	1913

t	5	6	7	8	8.824
x'	374.7	353.3	336.2	322.4	312.9
y'	−7.8	−16.9	−25.7	−34.2	−41.1
E	0.0636	0.0540	0.0453	0.0388	0.0344
$\frac{v^2}{100}$	1404	1251	1137	1051	996.1
x	2301	2664	3009	3338	3599

Questions on Chapter XIV.

1. Why are the times, tabulated in the first row of the computing form, designated by t_0 instead of t_Δ or t? Why by t in problem 63?
2. Why the minus sign in the terminal value of μ? Compare equations 46 of Chapter IX.
3. Why is no first difference formed between the first two columns of computations?
4. How is the first part of the computation smoothed out?
5. The values from what columns are the range-table values?
6. For what are the values from the other columns used?
7. What difference in methods or algebraic signs, from those of Chapter VIII, is used to compensate for the fact that in the present chapter we are integrating from T to earlier times?
8. Suppose a tabular function for times 4 seconds, 3 seconds, and 2 seconds, i. e, f_4, f_3, and f_2. What is the value of a_3? What would this have been in the notation of Chapter VIII?

CHAPTER XV.

WEIGHTING FACTORS.

In considering how weighting factors are derived from the computations described in the preceding chapter, let us take as an example the weighting factors for range wind.

The computation sheet furnishes, among other data, the effect of a 1 m/s wind on the range. This is entered in the range table, so that a battery commander, by multiplying this by the number of meters per second in the range component of the wind, can calculate the wind correction to apply to his map range.

But this step by the battery commander is based on the assumption of a *uniform* wind at all altitudes, which is a condition that seldom, if ever, exists. The wind constantly changes both its velocity and direction from one altitude to another. Velocity usually increases with altitude. Accordingly, some means must be devised for calculating the velocity and azimuth of a purely *fictitious* uniform wind which would have an effect on range equal to the combined effect of all the actual winds met by the projectile in its flight. This fictitious wind is used by the battery commander just as though it were the actual wind. It is called the "ballistic wind." The British call it the "equivalent uniform wind," which is a very apt name.

In the field, the meteorological service measures the average direction and velocity of the actual wind for successive strata of 250 meters each.

Our problem is now to determine, for any given trajectory, what weight is to be given to the wind of each stratum in making up the ballistic wind. This is done by determining what proportional part each stratum plays in making up the total correction for a 1 m/s wind on the computation sheet of the preceding chapter. The procedure is as follows:

From the trajectory sheet of the original trajectory computations, find the time corresponding to $y'=0$, and the maximum ordinate (i. e., the value of y corresponding to this time,) by methods analogous to those used in getting the terminal values:[1]

$$\textbf{97.}\qquad \begin{cases} \Delta t = \dfrac{-y'}{a+\dfrac{\Delta t+1}{2!}b+\dfrac{(\Delta t+1)(\Delta t+2)}{3!}c+\cdots} \\[2ex] y_s = y+\Delta t a+\dfrac{\Delta t(\Delta t+1)}{2!}b+\dfrac{\Delta t(\Delta t+1)(\Delta t+2)}{3!}c+\cdots \end{cases}$$

[1] The Technical Staff practice is to use the method for slide rule given in footnote, Chapter VIII, for finding terminal conditions, adapting it to the present conditions.

taking the values of y' and its a, b, c, etc., and of y and its a, b, c, etc., from the line whose y' is nearest to zero. In the first equation, use the a, b, c, etc., of y' and solve by successive approximations, taking for a first approximation $\Delta t = \frac{-y'}{a}$. In the second equation, use the a, b, c, etc., of y. The result is y_s, the maximum ordinate.

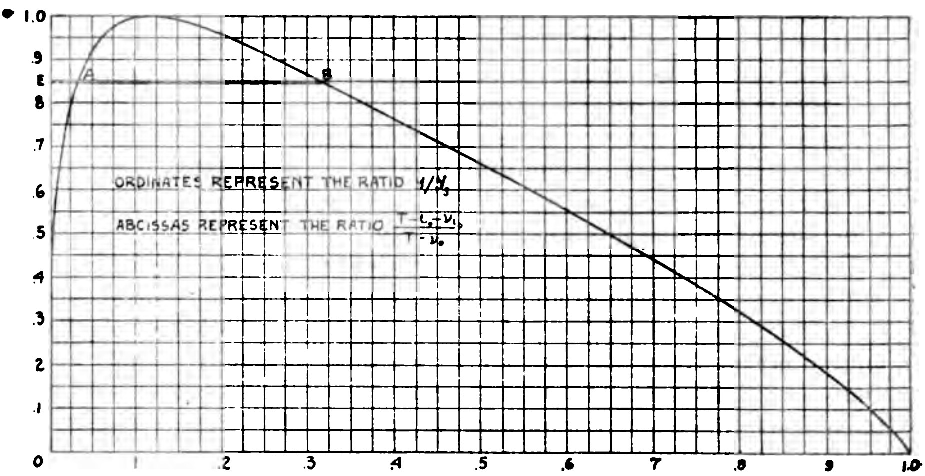


FIG 12

Against t tabulate y from the trajectory sheet and $T-t_0-\nu_{t_0}$ from line 53 of the differential computation sheet, and calculate $\frac{y}{y_s}$ and $\frac{T-t_0-\nu_{t_0}}{T-\nu_0}$, as follows:

t_0	T			2	1	0
y	0					0
$T-t_0-\nu_{t_0}$	0					$T-t_0$
$\frac{y}{y_s}$	0					0
$\frac{T-t_0-\nu_{t_0}}{T-\nu_0}$	0					1

This tabulation gives for each value of t_0: The altitude of the projectile; the range effect of a 1 m/s wind blowing from time t_0 to time T; the ratio of the altitude to the maximum ordinate; and the ratio of the aforementioned wind effect to that of a 1 m/s wind blowing throughout the entire flight of the projectile.

Now plot, on a small sheet of coordinate paper, $\frac{T-t_0-\nu_{t_0}}{T-\nu_0}$ against $\frac{y}{y_s}$, and connect the points by a smooth curve. (See fig. 12.) This curve will run from the point (1, 0), representing time zero, to the origin, representing time T; and will be tangent at its summit to the line $\frac{y}{y_s}=1$.

Let AB be drawn parallel to the axis of abscissas, a distance of *any* $\frac{y}{y_s}$ above it. Then EB represents the proportional effect of a 1 m/s wind blowing from the time the projectile first reaches altitude y, until the point of fall. EA represents the proportional effect of a 1 m/s wind blowing from the time the projectile reaches altitude y in its descent until the point of fall. Therefore AB represents the proportional effect of a 1 m/s wind blowing *above* the altitude y.

On a second sheet of coordinate paper, plot AB (measured to the left from the line of unit abscissas) against $\frac{y}{y_s}$. (See fig. 13.) Then FB equals $1-AB$, equals the proportional effect of a 1 m/s wind blowing *below* altitude y.

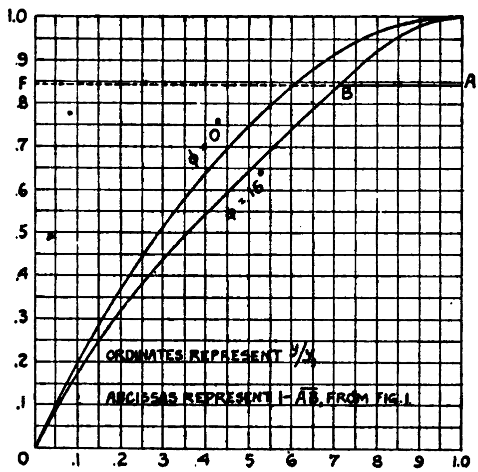


FIG. 13

The proportional effect of a wind blowing throughout any stratum is the difference between the FB corresponding to the top y of the stratum, and the FB corresponding to the bottom y of the stratum.

Thus, to get the weighting factors for the trajectory in question, mark off on the curve like that of figure 13, points whose $\frac{y}{y_s}$ are respectively $\frac{0}{y_s}$, $\frac{250}{y_s}$, $\frac{500}{y_s}$, $\frac{750}{y_s}$, etc. The difference in abscissas between the first and second points is the weighting factor for the first stratum; the difference in abscissas between the second and third points is the weighting factor for the second stratum, etc. As a check, the sum of all the weighting factors should be unity.

Of course, it would be possible to calculate the weighting factors directly from the curve like that of figure 12, but it is more convenient to have the weighting-factor curves in the form of figure 13, as it is

usual to plot on the same sheet the curves for various angles of departure of the same gun.

It should be noted that, for any given trajectory, the weighting factors for making up a ballistic wind to use for *range* corrections are quite different from the weighting factors for making up a ballistic wind to use in *deflection* corrections. Therefore, according to present methods, there are for each combination of C, V, and ϕ, *two* ballistic winds: (*a*) The *range* ballistic wind, whose range component is used as a basis for range corrections; and (*b*) the *cross* ballistic wind, whose lateral component is used as a basis for deflection corrections.

The foregoing method for getting weighting factors for range ballistic wind is equally applicable to cross ballistic wind and ballistic density, the last requiring slight modifications of a fairly obvious sort in referring changes aloft to equivalent changes at the ground. The treatment of ballistic elasticity has not yet been decided upon.

For each combination of C and V, a chart can be made up showing (for instance) the weighting factor curves for range ballistic wind for four or five values of ϕ. The curves for other values of ϕ can be interpolated, if only we have a curve for $\phi=0$. But of course no trajectory can be computed for $\phi=0$. Nevertheless, there has been deduced the equation of the limiting curve which the range wind weighting-factor curve approaches as ϕ approaches zero. The equation is:

98. $$1-p=\frac{3n}{4n-1}(1-k)^{1/2}+\frac{n-1}{4n-1}(1-k)^{3/2}$$

in which p is the abscissa and k the ordinate of any point on the curve; and n is the Gâvre n, to be obtained from the table of $(n-2)$, entered with $\frac{V^2}{100}$, using the standard muzzle velocity.

The equation of the limiting curve for density weighting factors is:

99. $$1-p=\frac{3}{4}(1-k)^{1/2}+\frac{1}{4}(1-k)^{3/2}$$

The equation of the limiting curve for cross wind weighting factors is·

100. $$1-p=(1-k)^{1/2}$$

The limiting curve for elasticity weighting factors is the same as that for density; but, owing to the fact that elasticity weighting factors become infinite in certain cases, the use of such weighting factors is not to be recommended.

It will be observed that in the case of density, elasticity, and cross wind, the limiting curve is independent of muzzle velocity and ballistic coefficient. In the case of range wind the limiting curve depends on the muzzle velocity, but not on the ballistic coefficient.

The weighting-factor curves for $\phi = 0$ can be plotted on the weighting-factor charts with very little extra work, and will assist materially in the interpolation of new curves among those computed.

By comparing a large collection of weighting-factor curves at Aberdeen, three mean wind weighting-factor curves have been deduced, whose equations are, respectively:

$$101. \qquad \begin{cases} 1-p=1.11\ (1-k)^{1/2}-0.11\ (1-k)^{2} \\ 1-p=0.74\ (1-k)^{1/2}+0.26\ (1-k)^{2} \\ 1-p=0.36\ (1-k)^{1/2}+0.64\ (1-k)^{2} \end{cases}$$

For any given battery, one of these curves can be chosen, which will give a sufficiently approximate ballistic wind for both range and deflection corrections for actual field service.

Similarly the following single density weighting-factor curve has been deduced:

$$102. \qquad 1-p=0.48\ (1-k)^{1/2}+0.52\ (1-k)^{3/2}$$

Prior to the World War a single curve, known as the "time curve" or "vacuum curve" was used for all weighting-factor purposes. These names are due to the fact that this curve weights the various strata of atmosphere in proportion to the time the projectile would spend in each, if the trajectory occurred in vacuo. Its equation was the same as that of the limiting curve for cross wind (equation 100).

The present indications are that the second equation under 101 will be adopted by both branches of artillery[2] for all wind weighting-factor purposes; and equation 102 for all density-weighting factor purposes.

PROBLEMS.

(64) The maximum ordinate of problem 63 was $y_s=97.9$. Construct the range-wind and the cross-wind weighting factor curves on the same sheet of cross-section paper.

(65) Plot equations 102 and 100 on a single sheet for comparison of the old and the modern mean density weighting-factor curves.

(66) Plot equations 101 and 100 on a single sheet for comparison of the old and the modern mean wind weighting-factor curves.

QUESTIONS ON CHAPTER XV.

1. Define "ballistic wind."
2. What is the British term for "ballistic wind?"
3. What are "weighting factors?"
4. Explain the two ballistic winds used in present-day methods? Are these the components of a single ballistic wind?
5. As no trajectory can be computed for $\phi = 0$, how are the weighting factors obtained for this angle of departure?
6. Of what use are the curves for $\phi = 0$?
7. Are weighting factor curves used to determine ballistic elasticity?
8. Define "ballistic density."
9. What is meant by the "time curve"?

[2] As a result of experiments at Fort Monroe, the Coast Artillery has adopted the *second* of equations 101 for all wind purposes.

CHAPTER XVI.

CONSTRUCTION OF A RANGE TABLE.

A range firing consists of a number of rounds, usually 10 to 20, fired at each of several elevations, say, at 5°, 15°, 25°, 35°, and 45°, The exact location of each point of splash is plotted by means of intersecting azimuths from at least four observation towers.

Throughout the firing the meteorological conditions are observed from time to time by aeroplane, pilot balloon, etc. At the gun a detailed record is kept of the time of firing, quadrant elevation, and of all variations from standard, such as weight of projectile, cant of trunnions, etc. Time of flight is taken by stop watch as a check on the computations later to be made and as a basis for the rough determination of maximum ordinate ($y_s = 4.05\ T^2$).

Separate rounds are usually fired through jump and chronograph screens, as a basis for determining jump and muzzle velocity.

Given the range-firing records for a specified gun and projectile, with prescribed values of weight of projectile and muzzle velocity, to construct a range table for this gun and projectile, the procedure is as follows:

Divide the total rounds fired into groups, each having identical ranges (except for slight differences due to dispersion) because of having been fired at the same elevation and on the same day.

Compute, for each group, the mean values of all measured quantities. These measured quantities are muzzle velocity, weight of projectile, observed range, deflection, right wheel above left, time of flight, etc.

With average values of the atmospheric conditions, and with weighting factors from a similar range table, compute a tentative ballistic range wind, cross wind, temperature, and density. If no similar range table is available, use the mean weighting-factor curves of Chapter XV.

Compute, for each group, the angle of departure, by correcting the quadrant elevation for any individual error in the quadrant, for any inclination between quadrant seat and axis of the bore (determined by applying a clinometer to the gun) for jump and for height of site. (See Chap. XI.)

Unless velocities were taken on the range rounds themselves, estimate the mean muzzle velocity of each range group, as follows: Let δp be obtained by substracting from the mean projectile weight of the group in question, the mean projectile weight of the velocity

rounds fired on the same day. Then the estimated velocity of the range group will be the algebraic sum of δV and the mean velocity of these velocity rounds, δV being obtained from the equation:

$$\frac{\delta V}{V} = n\frac{\delta p}{p}.$$

See the explanation preceding formula 58.

Estimate the ballistic coefficient for each group. This may be done in a number of ways. The observed ranges should first be roughly corrected for nonstandard conditions, using a similar range table, or the Ingalls tables, or Alger's charts (which are a graphic representation of the Ingalls correction formulas), or French charts, or the A. L. V. F. tables, or the Gâvre tables of September 15, 1917. The last-named two are French tables. An Americanization of the A. L. V. F. tables has been published by the Ordnance Department (War Department, Document No. 983, *Confidential*), and an Americanization of the Gâvre tables by the Coast Artillery Board (mimeographed—title: "Artillery Ballistic Tables").

Then with ϕ, V, and the corrected X as arguments, enter "Ingalls' Ballistic Tables" (printed by War Department, 1918), or a set of French charts, or a set of Alger's charts (Journal of the U. S. Artillery, Dec. 1919, p. 585), and take out C. It should be noted that the C obtained by either of these methods is not the C desired. The C of the Ingalls tables and the Alger charts is the Siacci C, so-called:

103. $$C_s = \frac{wf_a}{\beta i d^2}$$

The C of trajectory computations in modern ballistic methods is the normal C, so-called:

104. $$C_N = \frac{w}{id^2}.$$

The coefficient of form, (i) of the two C's is also slightly different. But a tentative value for C_N can be obtained by extracting β and f_a[1] from C_s. The use of French tables or charts is, however, much better.[1]

105. $$\begin{cases} \beta = \sqrt{\sec\phi} \\ \log_{10}\left(\dfrac{1}{f_a}\right) = -0.00012\ T^2. \end{cases}$$

[1] $\frac{1}{f_a}$ is the mean value of H for the trajectory. This is assumed to be the same as the H at an altitude two-thirds the maximum ordinate. The Ingalls formula for maximum ordinate in terms of time of flight is $y_s = 4.05\ T^2$. Therefore:

$$\log_{10}\left(\frac{1}{f_a}\right) = \log_{10} H = -0.000045\left(\frac{2y_s}{3}\right) = -0.00003\ y_s = -0.0001215\ T^2.$$

The French C and the normal C are bound together by the following approximate relation:

106.
$$\begin{cases}\log_{10} C_F + \log_{10} C_N = 7.0570 - 10 \\ C_F \ . \ C_N = 0.001140.\end{cases}$$

The Americanized French tables give C_N direct, instead of C_F.

Plot the approximate values of C thus found against ϕ, and draw a smooth curve. For each value of ϕ at which firings were made, take the smoothed-out value of C and the standard V and compute a trajectory by the methods laid down in Chapter VIII.

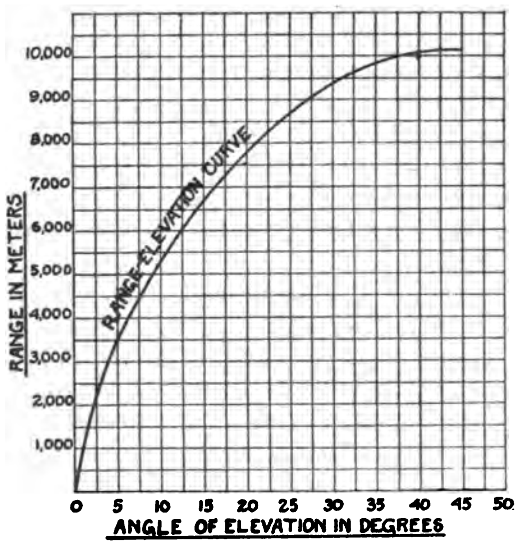


FIG. 14

Compute the differential corrections by the methods of Chapter XIV for each of the computed trajectories.

The variations from standard for which the differential corrections are to be computed are as follows:

1 m/s change in muzzle velocity.
1 per cent change in atmospheric density.
1 per cent change in absolute temperature (as affecting elasticity).
1 m/s range wind.
1 mil change in elevation.
1 per cent change in weight of projectile.[2]
1 m/s cross wind.

From these results, compute the changes in range for the variation of actual conditions from standard (atmosphere, muzzle velocity, weight of projectile, and elevation) at the time of firing, and correct the observed ranges to standard ranges (i. e., to the ranges which the given ϕ and V, and the estimated C would have produced under standard conditions).

---

[2] Only the first term of formula 58 should be used in getting the standard range from the observed range. In getting the estimated velocity of the range rounds from the observed velocity rounds, use $\frac{\delta V}{V} = n \frac{\delta p}{p}$ as before. (See the explanation which precedes formula 58.)

Compare these corrected ranges with the ranges obtained from the trajectory computations. If the differences are considerable, say, greater than 5 per cent, correct the values of the ballistic coefficients at which the trajectories were computed and recompute the trajectories and the differential corrections for the new values of C. Then repeat the work indicated in the preceding paragraph.

The range-elevation curve should now be constructed. Correct each quadrant elevation (i. e., observed elevation) by adding the angle of site and the corrections for the quadrant and quadrant seat. We do not here correct for jump, as what is here wanted is the angle of elevation rather than the angle of departure. Plot the corrected ranges against the corresponding elevations as thus obtained. (See fig. 14.)

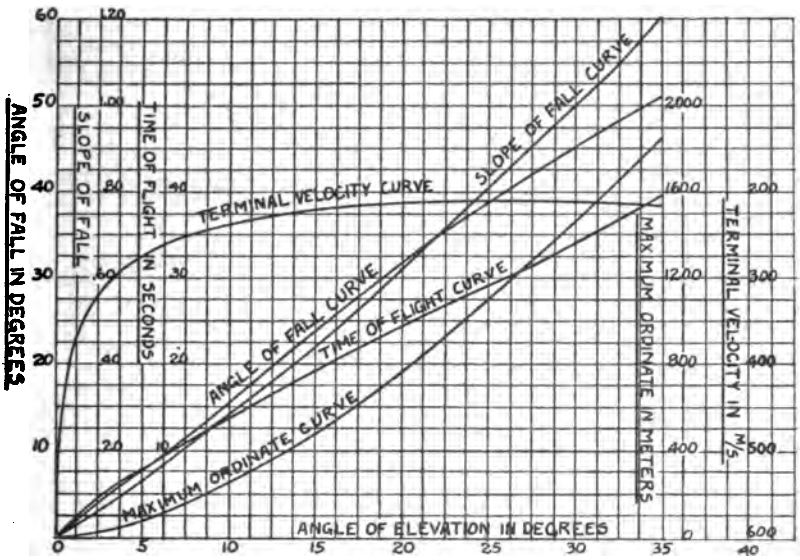


FIG. 15

Certain elements of the trajectory and certain differential corrections should now be plotted against the angle of elevation corresponding to the angle of departure from which they were computed. (See fig. 15.) On one sheet of cross-section paper should be plotted—

Slope of fall.
Angle of fall (ω).
Time of flight (T).[3]
Terminal velocity (v_T).
Maximum ordinate (y_s).

[3] Both the computed and the observed times of flight are plotted. The curve is then drawn, giving by far the greater weight to the computed values, but giving some weight to the observed.

On one sheet should be plotted (See Fig. 16) the range changes due to—

1 m/s increase in muzzle velocity.

1 per cent decrease in atmospheric density.

1 per cent increase in absolute temperature (only as affecting elasticity).

1 m/s following wind.

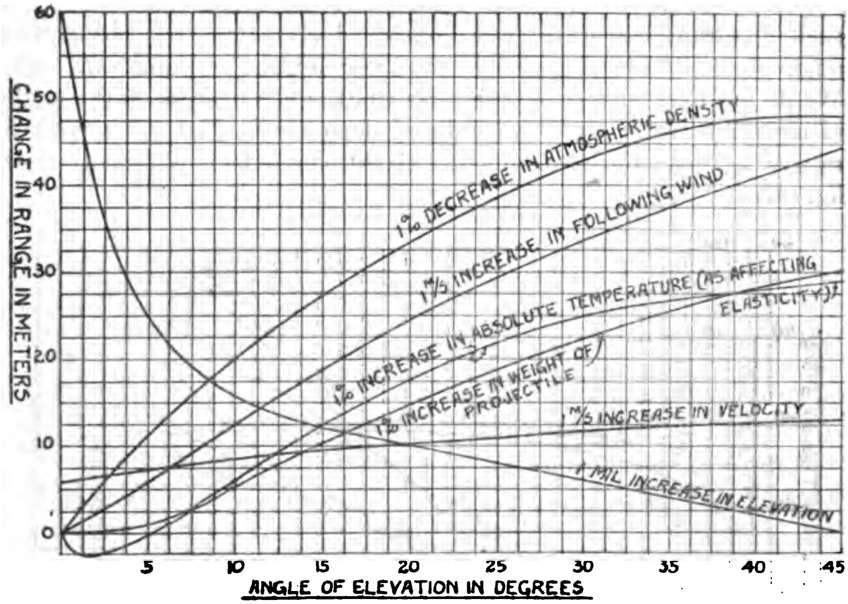


FIG 16

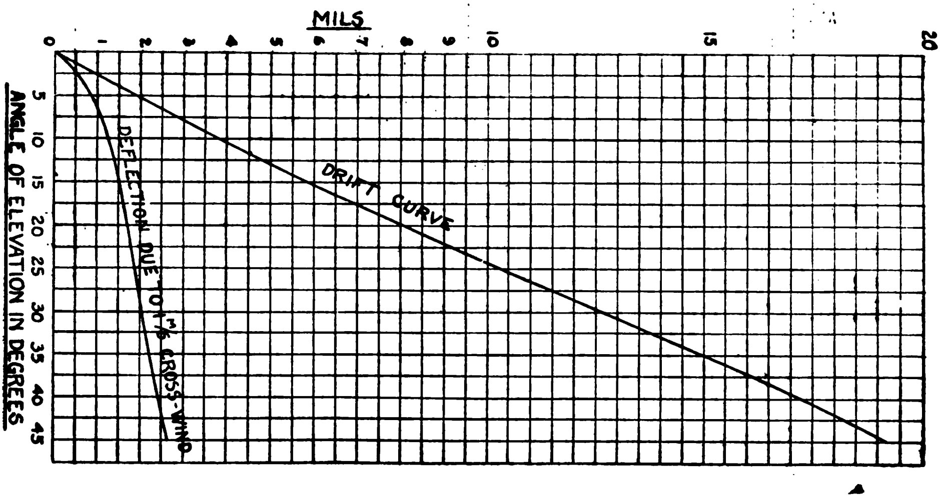


FIG. 17

1 mil increase in elevation.

1 per cent increase in weight of projectile.[4]

On one sheet should be plotted (see fig. 17) the deflection effect, in mils, of—

1 m/s cross wind.

Drift (including lateral jump).

Compute the probable error for range and deflection. This is obtained by multiplying the mean deviation by 0.845. Usually 20 or more rounds are fired at each of several elevations for the determination of dispersion alone. The dispersion of the range groups is also considered if these contain enough rounds. Plot the 50 per cent zones against elevations. The 50 per cent zone is twice the probable error.

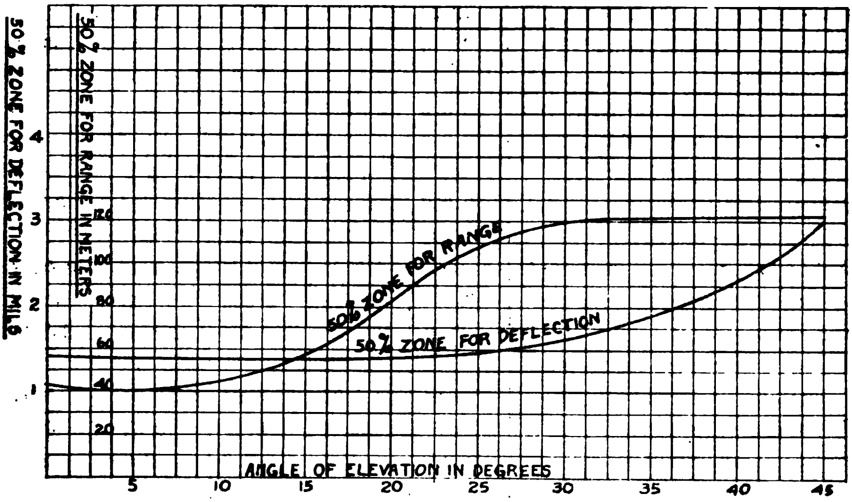


FIG. 18

Any additional data which may be needed should be treated in the same way.

All curves should be smoothed out. Practically no smoothing should be necessary, except in the case of the drift curve and the probable error curves.

From these curves take the necessary data for the construction of the range table.

Typical curves are shown in figures 14 to 18, inclusive, which are based on the computation of a range table for the 75 mm gun, firing a Mark IV projectile at 1,900 f/s.

[4] The whole of formula 58 should be used here.

PROBLEM.

(67) Construct the necessary curves from the following data:

155 mm G. P. F. gun firing 94.7-pound shell at 2,410 f/s velocity.

	Date.				
	June 9.	June 9.	June 9.	June 10.	June 14.
Number of rounds considered	25	13	12	20	20
Clinometer elevation	6° 4′	16° 35′	16° 35′	25° 5′	35° 5′
Range (meters)	6,477	11,288	11,453	14,142	16,564
Deflection (mils)	5.9	10.7	8.1	6.3	7.4
Time of flight (seconds.)	13.23	29.85	30.06	41.89	55.0
Weight of projectile (pounds)	93.48	93.67	93.53	93.52	93.37
Mean deviation in range (meters)	44	54	61	107	94
Mean deviation in deflection (mils)	0.5	0.5	0.6	0.7	0.6
Ballistic range wind (m/s)	+2.3	+3.4	+3.7	−5.4	−8.8
Ballistic cross wind (m/s)	+1.8	+2.5	+0.6	−2.9	−9.0
Ballistic density	0.978	0.982	0.981	0.972	0.965
Ballistic temperature (°F)	70°	70°	70°	65°	71°
Right wheel above left (meters)	−0.0033	−0.0027	−0.0027	−0.0003	−0.0027
Distance between wheels (meters)	0.503	0.503	0.503	0.503	0.503
Angle of bore sight	6° 4′	16° 35′	16° 35′	25° 5′	35° 5′
Height of trunnions above mean low water (meters)	5.94	5.94	5.94	5.94	5.94
Height of tide (meters)	0.54	0.40	0.40	0.61	0.43
Vertical jump (minutes)	−2.0	−2.0	−2.0	−2.0	−2.0
Weight of projectile for velocity rounds (pounds)	93.33	93.33	93.33	93.35	95.64
Velocity of velocity rounds (f/s)	2,391	2,391	2,391	2,401	2,410

QUESTIONS ON CHAPTER XVI.

1. Give a very general outline of the steps in the construction of a range table.
2. How may the ballistic coefficient be estimated?
3. Why do you suppose y_s equals 4.05 T^2, instead of 4.02 T^2?
4. In constructing the range-elevation curve, why is not the observed elevation corrected for jump?
5. To convert the observed range to the standard range, should the results of solving the formulas of Chapter X be added or subtracted? Why?
6. What curves are likely to need smoothing out, and why?

SUPPLEMENT A.

TRAJECTORY COMPUTATION BY THE TANGENT RECIPROCAL METHOD.

The tangent reciprocal method of computing trajectories is a variant of the rectangular method described in Chapter VIII. It is based on the following equations, involving three auxiliary variables σ, σ', and σ'':

107. $$\begin{cases} \sigma = \dfrac{y'}{x'g} = -\dfrac{y'\sigma'}{g} & \text{(i. e., } \tan\theta \text{ divided by } g) \\ \sigma' = -\dfrac{1}{x'} & \text{(i. e., minus the reciprocal of the horizontal component of velocity)} \\ \sigma'' = E\sigma' & \end{cases}$$

These three variables are "critically varying", i. e., they change at such rates that inaccuracies produce the least possible effect on the results. This enables the use of longer time intervals with the tangent reciprocal method than with the original rectangular method.

For the tangent reciprocal method, change the trajectory sheet as follows: Use the $-Ex'$ column for y. Use the velocity, mean height, and time columns, and the first half of the $\int y\,dt$ column for y'. Use the last half of this column for time. Use the y column for σ, the y' column for σ', and the $-Ey'-g$ column for σ''.

On the small computing sheet, label the rows as follows (the numerals show the order in which the rows are used):

t	0	1	2	Order.
log G				8
log H				9
colog C				Constant.
log E				10
log σ'				1
log σ''				11
colog σ'				2
log g				Constant.
log σ				3
log y'				4
$x'^2/100$				5
$y'^2/100$				6
$v^2/100$				7

The logarithms are all denary.

To start the computation, enter on the trajectory sheet following initial values:

$$
\begin{aligned}
x &= 0 \\
x' &= V \cos \phi \\
y &= 0 \\
y' &= V \sin \phi \\
\sigma' &= -\frac{1}{x'} \\
\sigma &= -\frac{y'\sigma'}{g}
\end{aligned}
$$

Enter on the small sheet log $H=0$, for $t=0$; colog C (constant throughout); log g (constant throughout). Take $\frac{x'^2}{100}$ and $\frac{y'^2}{100}$ from the table of squares; add them to get $\frac{v^2}{100}$; and with this as an argument, get log G from the G Table.

Add log G, log H, and colog C to get log E. Add log E to log σ' to get log σ''. Enter σ'' on the trajectory sheet. Note that initially σ is plus, and σ' and σ'' are minus. This completes the computation for $t=0$.

To start a new line, integrate σ'' ahead to get the increment of σ'. Integrate σ' to get the increment of σ.

Turn to the small sheet. Set down log σ'. From this, get colog σ'. Set down log σ. Add colog σ', log g and log σ to get log y'.

Enter the table of logs and squares with log x' (same as colog σ') and log y' as arguments, and take out x', y', $\frac{x'^2}{100}$ and $\frac{y'^2}{100}$. Add the last two to get $\frac{v^2}{100}$. With this enter the G table and take out log G.

Integrate y' to get the increment of y; and, with y as an argument, get log H from the formula

$$\log H = (10 - 0.000045y) - 10.$$

Add log G, log H and colog C, to get log E. Add log E and log σ' to get log σ''.

Integrate σ'' to get the increment of σ', and σ' to get the increment of σ. If these new values check sufficiently close, the work on that line is completed.

Proceed in the same way with each successive line.

SUPPLEMENT B.

EXPLANATION OF THE SIGNS IN THE COMPUTATION OF DIFFERENTIAL CORRECTIONS.

Whatever may be their physical significance, all of the integrations of Chapter XIV are, mathematically speaking, performed from a later to an earlier time.

In integrating from a later to an earlier time, the first differences of the integral must receive an algebraic sign opposite to that of the integrand; in other words, a positive integrand produces an algebraically decreasing integral and vice versa. Thus the integral: $\int_{T}^{t_0}(\quad)\,dt_\Delta$ requires that a change of sign be made when numerically integrating; the integral: $-\int_{T}^{t_0}(\quad)\,dt_\Delta$ does not. But it is bad psychology to use a notation in which plus requires a change of sign, and minus does not; hence, although integrating backward, we shall use the symb l $\int_{t_0}^{T}$ in place of the symbol $\int_{T}^{t_0}$. Then, since

$$+\int_{T}^{t_0}(\quad)\,dt_\Delta=-\int_{t_0}^{T}(\quad)\,dt_\Delta, \text{ and}$$

$$-\int_{T}^{t_0}(\quad)\,dt_\Delta=+\int_{t_0}^{T}(\quad)\,dt_\Delta,$$

a minus will mean to change signs, and a plus will mean not to change.

Chapter XIV employs two sorts of integrals. The integrations to get μ' and μ are of one sort. Since μ' is the derivative of μ then, μ is the *indefinite* integral of μ'. That is—

$$\mu_{t_0}=\int^{t_0}\mu'dt_\Delta.$$

But the values of μ are fixed by being known at time T. Hence:

$$\mu_{t_0}=\int^{t_0}\mu\,dt_\Delta=\mu_T+\int_{T}^{t_0}\mu\,dt_\Delta=\mu_T-\int_{t_0}^{T}\mu\,dt_\Delta.$$

in line 22 of the computation sheet.

This equation means that μ is the indefinite integral of μ'; that its value is known at time T; that to find its value for an earlier time, one integrates from T to this earlier time; and that one changes sign in integrating. Similarly:

$$\mu'_{t_0}=\int^{t_0}\mu''dt_\Delta=\mu'_T+\int_{T}^{t_0}\mu''dt_\Delta=\mu'_T-\int_{t_0}^{T}\mu''dt_\Delta.$$

The other integrals of Chapter XIV are of a different sort, namely *definite* integrals from time t_0 to time T, representing the effects, at

time T, of causes starting at time t_0. Thus, in line 40 of the computation sheet:

$$(8) = +\int_{t_0}^{T} (7)\ dt_\Delta.$$

Logically we ought to integrate from each t_0 to T, proceeding from right to left, as follows:

t_0.	16.760	16	14	12	11	10
(7)........	0	0.08	0.30	0.52	0.62	0.73
a........	....	−0.22	−0.22	−0.10	−0.11	....
$(8) = +\int_{16}^{t_0}(7)dt_\Delta$........	0.03	0	....	....	....	....
$(8) = +\int_{14}^{t_0}(7)dt_\Delta$........	0.41	0.38	0	....	....	....
$(8) = +\int_{12}^{t_0}(7)dt_\Delta$........	1.23	1.20	0.82	0	....	....
$(8) = +\int_{11}^{t_0}(7)dt_\Delta$........	1.80	1.77	1.39	0.57	0	....
$(8) = +\int_{10}^{t_0}(7)dt_\Delta$........	2.47	2.44	2.06	1.24	0.67	0
Etc.	Etc.	Etc.	Etc.	Etc.	Etc.	Etc.

But, if we are willing to give up logical arrangement, in order to save labor and space, we can obtain the same results as follows, integrating from left to right:

t_0.	16.760	16	14	12	11	10
(7)........	0	0.08	0.30	0.52	0.62	0.73
a........	....	....	0.22	0.22	0.10	0.11
$(8) = +\int_{t_0}^{T}(7)dt_\Delta$........	0	0.03	0.41	1.23	1.80	2.47

In either way, the plus sign in front of the integral signifies that *no* change of sign is made in integrating.

The heavy black line between two columns is a convenient device to indicate a change in time interval.

To recapitulate, although all integrations in Chapter XIV are made from time T to time t_0, we shall use the integral symbol $\pm\int_{t_0}^{T}$, so that a minus sign before the integral shall serve notice that the increment of the integral is to be given a sign opposite to the sign of the integrand, and so that a plus sign shall serve notice that no such change of signs is to be made.

SUPPLEMENT C.

DIMENSIONS OF BALLISTIC SYMBOLS.

In deriving ballistic formulas, it is frequently convenient to test them by the theory of dimensions. For this purpose we shall employ the following three dimensions: Length (L), time (T), and mass (M). Unity will be expressed by 1.

The following are the dimensions of the principal ballistic symbols:

Symbol.	Dimensions.
x and y	L
x', y', and v	L/T
x'', y'', and g	L/T^2
G	L/T
H	M/L^3
h	$1/L$
C	M/L^2
w	M
i	1
$\dfrac{d \log G}{v\,dv}$	T^2/L^2
$10^6\left(h+g\,\dfrac{d \log G}{v\,dv}\right)$	$1/L$
E	$1/T$
μ	L
μ'	L/T
μ''	L/T^2
ν and ρ	T
ν' and ρ'	1
All trigonometric functions	1
All angles	1
All logarithms	1
All exponents	1

All of the foregoing dimensions are fixed by physical laws except the dimensions of G, H, and C.

There is a wide latitude possible in choosing the dimensions for these three symbols, provided only that the dimensions of $\frac{GH}{C}$ are $\frac{1}{T}$.

Thus C may be treated as a dimensionless constant and H a dimensionless ratio; G will then have then same dimensions as E, which convention would have much to commend it. Or vG could be regarded as having the dimensions of its Mayevski equivalent $A_n v^n$, i. e., $\frac{L^n}{T^n}$, whence G would have the dimensions $\frac{L^{n-1}}{T^{n-1}}$.

The dimensions actually adopted for G, H, and C were arrived at as follows: G was regarded as $vB\left(\frac{v}{s}\right)$, $\frac{v}{s}$ being treated as a dimensionless ratio (see equations 59 to 64 in Chap. X). H was given the dimensions of density, and C the dimensions of sectional density. The following physical laws were thus expressed:

$$G=vB\left(\frac{v}{s}\right)$$

$$H=\frac{\delta}{1203.4}$$

$$C=\frac{w}{id^2}$$

The usefulness of the tabulated dimensions is as follows: To test the dimensional correctness of a formula, substitute for each symbol its dimensions, disregarding algebraic signs and non-dimensional numerical coefficients. If the dimensions of all the terms are alike, it is dimensionally correct. Very often typographical errors and mistakes in derivation can thus be readily detected.

Supplement D.

ANTIAIRCRAFT FIRE.

At the time of writing this book the methods of computing differential corrections for antiaircraft fire are in such an unsettled state that it is thought best not to describe them in detail, but merely to refer to them in a general way.

In Chapter IX we saw that the use of three auxiliary variables, μ, ν, and ρ, and an auxiliary constant, λ, enabled us to express a change in X, T, or y_s in terms of changes in x, y, x', and y' occurring at any time, the auxiliary variables being functions of the time of the disturbance. The expressions for δX, δT, and δy_s in that chapter may be generalized into

108. $$\delta(\quad) = \lambda\,\delta x + h\mu\,\delta y + \nu\,\delta x' + \rho\,\delta y'$$

where the parenthesis represents an effect of any given nature occurring at any given point on the trajectory. Thus δX is the effect on x at the point of fall, δT is the effect on t at the point of fall, δy_s is the effect on y at the summit. Similarly we might use the same general form to express any given effect at any other point on the trajectory. The auxiliary variables here used are not to be confused with the special cases considered in Chapters IX, etc.

In the general form given above, δx, δy, $\delta x'$, and $\delta y'$ are arbitrary small increments occurring at the time of the cause (which we shall call t_Δ).

x, y, x', y', x'', and y'' are functions of the trajectory and of the time (t) of the point on the trajectory at which their value is taken, whether this be the time of cause ($t = t_\Delta$), or the time of effect ($t = t_e$), or some other time.

λ, μ, ν, and ρ are functions of the trajectory, of the nature of the effect considered, of the time of cause (t_Δ), and of the time of effect (t_e).

$\delta(\quad)$ occurs at time t_e and is a function of the trajectory, of the nature of the effect considered, and of δx, δy, $\delta x'$, $\delta y'$, t_Δ, and t_e.

Let us now consider some particular trajectory and some particular effect, such as change in x. Let us successively consider some fixed value of t_e, the time of effect, so that it successively equals 0, 2, 4, 6 . . . T.

Then for each value of t_e:

109. $$\delta x_{t_e} = \lambda\,\delta x + h\mu\,\delta y + \nu\,\delta x' + \rho\,\delta y'.$$

For each value of t_e, we can get a different set of values of λ, μ, ν, and ρ as functions of t_Δ alone.

The entire computation of Chapter XIV is in effect repeated for each value of t_e, running each computation backward from $t_0 = t_e$ to $t_0 = 0$, the result being a tabulation of differential corrections, from which can be determined the effect on the x coordinate of a projectile at any point in its flight, due to a disturbance at any preceding point or points in its flight.

The terminal values used to start each of these computations are:

110. $$\begin{cases} \mu = -\dfrac{\cot \theta}{h} \\ \mu' = 0 \\ \mu'' = 0 \\ \nu = 0 \\ \rho = 0. \end{cases}$$

This is a very tedious performance. Accordingly there have been devised several alternative methods, each involving variables auxiliary to the auxiliary variables, i. e., bearing much the same relation to the auxiliary variables as the auxiliary variables do to the elements of the trajectory.

In the latest method, two sets of values for μ and μ' are computed, each based on the assumption that equation 108 holds true and that $\lambda = 0$; one based on the assumption that $h\mu$ is zero and μ' unity at the gun, the other on the assumption that $h\mu$ is unity and μ' zero at the gun. These four variables, called μ_3, μ'_3, μ_4, and μ'_4, are functions of the time of cause alone. In this method the integration is performed forward.

Variable coefficients K_3, K_4, L_3, L_4, M_3, M_4, N_3, N_4, and Δ are also computed, these being functions of μ_3, μ_4, μ'_3, and μ'_4, and of the time of effect. This Δ is a quantity, and not an operator.

Attempts are now being made to simplify this system, Until this is accomplished, it is not thought advisable to present the matter to students in any more detail than is here given.

Supplement E.

DERIVATION OF TWO EQUATIONS OF CHAPTER VII.

The precise equations of motion of a projectile in the tangent method (equations 30) or the curved method (equations 31) have been derived in various blue prints of the Ordnance Department. A brief skeleton of the derivation is here given.

The equations of motion referred to an instantaneous system of cartesian axes,[1] whose origin is the instantaneous position of the projectile, and which are respectively horizontal and vertical at that point, are:

$$\text{111.} \qquad \begin{cases} \bar{x}'' = -E\bar{x}' \\ \bar{y}'' = -E\bar{y}' - g \\ v^2 = \bar{x}'^2 + \bar{y}'^2 \\ \tan\theta = \dfrac{\bar{y}'}{\bar{x}'} \end{cases}$$

The tangent method assumes, in place of an infinite number of sets of instantaneous axes, a single set, namely, the instantaneous set whose origin is the gun. The only effect of this assumption is to change the basis of the H function and the manner in which gravity enters into the equations. Also, in *any* system, gravity (g) ought to be expressed in terms of the constant surface gravity (g_0).

If ψ is the angle at the center of the earth subtended by the flight of the projectile up to its arrival at the point XY, if ρ is the distance from the center of the earth to the projectile and R the distance from the center of the earth to the gun, and if gravity varies inversely as ρ^2, then:

$$\text{112.} \qquad g = g_0\frac{R^2}{\rho^2} = g_0\frac{R^2\cos^2\psi}{(R+Y)^2} = g_0\left(1-\frac{2Y}{R}\;\cdots\right)\left(1-\;\cdots\right)$$

The components of gravity are:

$$\text{113.} \qquad \begin{cases} g_X = g\sin\psi = g_0\left(1-\dfrac{2Y}{R}+\;\cdots\right)\left(\dfrac{X}{R}-\;\cdots\right) \\ g_Y = g\cos\psi = g_0\left(1-\dfrac{2Y}{R}-\;\cdots\right)\left(1-\;\cdots\right) \end{cases}$$

Therefore the equations of motion are as given in equations 30 of Chapter VII.

[1] Coordinates $\bar{x}$ and $\bar{y}$ are those of the instantaneous system. Coordinates X and Y are those of the tangent method. Coordinates x and y are those of the curved method.

Also the correct basis of the H function becomes:

114. $$\rho-R=\frac{R+Y}{\cos\psi}-R=Y+\frac{X^2}{2R}\pm\ \cdots$$

The curved method assumes a single system of orthogonal curvilinear coordinates, whose x is measured from the gun along a circle concentric with the earth, and whose y is measured vertically upward from this circle.

The equations of motion of the instantaneous system are converted into polar coordinates ψ and ρ as defined above, and thence into the orthogonal curvilinear coordinates.

The first conversion is derived as follows: Consider a point P_1 on the trajectory near the instantaneous point P, and denote by $\Delta\ \psi$ the angle at the center of the earth subtended by PP_1. Then:

115. $$\begin{aligned}\bar{x}&=\rho\sin\Delta\ \psi\\ \bar{y}&=\rho\cos\Delta\ \psi-OP\end{aligned}$$

where O is the center of the earth.

Differentiate twice, with respect to time, and then let P_1 approach P. Cos $\Delta\ \psi$ will approach unity, and sin $\Delta\ \psi$ will approach zero, and we shall have as conversion equations:

116. $$\begin{cases}\bar{x}'=\rho\psi'\\ \bar{y}'=\rho'\\ \bar{x}''=2\rho'\psi'+\rho\psi''\\ \bar{y}''=\rho''-\rho\psi'^2\end{cases}$$

The relation between the polar coordinates (ψ, ρ) and the coordinates (x, y) of the curved system is:

117. $$\begin{cases}\psi=\dfrac{x}{R}\\ \rho=y+R\end{cases}$$

Differentiating twice, with respect to time, we get as conversion equations:

118. $$\begin{cases}\psi'=\dfrac{x'}{R}\\ \rho'=y'\\ \psi''=\dfrac{x''}{R}\\ \rho''=y''\end{cases}$$

Substituting from equations 118 and 119 in equations 117:

119. $$\begin{cases} \bar{x}' = x' + \dfrac{x'y}{R} \\ \bar{y}' = y' \\ \bar{x}'' = x''\left(1+\dfrac{y}{R}\right)+\dfrac{2x'y'}{R} \\ \bar{y}'' = y'' - \dfrac{x'^2}{R} \ldots \end{cases}$$

Substituting from equations 119 in equations 111, correcting g for altitude, but not for obliquity, and neglecting y in comparison with R, we get the equations of motion as given in equations 31 of Chapter VII.

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SUPPLEMENT F.

A DERIVATION OF THEOREM 1.

An inspection of any number of formulas of ordinary differentiation will show that they are homogeneous with respect to the ordinary differential operator d.[1] Therefore, let us assume, as is indeed the case, that there is no possible form of differentiation which violates this principle.

Let

$$\begin{aligned} u &= f_1\ (X, Y, Z, \ldots) \\ X &= f_2\ (x, y, z, \ldots) \\ Y &= f_3\ (x, y, z, \ldots) \\ Z &= f_4\ (x, y, z, \ldots) \end{aligned}$$

etc.

Then, by the foregoing assumption:

$$du = AdX + BdY + \ldots \qquad \textbf{(a)}$$
$$dX = Cdx + Ddy + \ldots \qquad \textbf{(b)}$$
$$dY = Edx + Fdy + \ldots \qquad \textbf{(c)}$$

where A, B, C, D, E, F, etc., are undetermined coefficients.

Substitution from equations **b**, **c**, etc., in equation **a** gives:

$$du = (AC + BE + \ldots)\ dx + (AD + BF + \ldots)\ dy + \ldots \qquad \textbf{(d)}$$

In equation **a**, u was considered as a function of X, Y, Z, etc. In equation **d**, u is considered as a function of x, y, z, etc.

Now divide equation **a** through by dX. Then:

$$\frac{du}{dx} = A + B\frac{dY}{dX} + \ldots$$

If, in this expression, Y, Z, etc., be regarded as constant, then $\frac{du}{dx}$ becomes $\frac{\partial u}{\partial x}$ by definition of the partial derivative, and all the other derivatives of this equation vanish. Thus A is identified as $\frac{\partial u_{XYZ\ldots}}{\partial x}$.

[1] This means that, if we treat d as an algebraic quantity, instead of as an operator, each term will contain d to the same power as any other term.

Similarly:

$$B=\frac{\partial u_{XYZ\ldots}}{\partial Y}$$

$$C=\frac{\partial X_{xyz\ldots}}{\partial x}$$

$$D=\frac{\partial X_{xyz\ldots}}{\partial y}$$

$$E=\frac{\partial Y_{xyz\ldots}}{\partial x}$$

$$F=\frac{\partial Y_{xyz\ldots}}{\partial y}$$

Now divide equation **d** through by dx, and regard y, z, etc., as constant. The equation thus becomes:

$$\frac{\partial u_{xyz}}{\partial x}=AC+BE+\cdots$$

Substitute the identified equivalents of A, C, B, E, etc. Then:

$$\frac{\partial u_{xyz\ldots}}{\partial x}=\frac{\partial u_{XYZ\ldots}}{\partial X}\cdot\frac{\partial X_{xyz\ldots}}{\partial x}+\frac{\partial u_{XYZ\ldots}}{\partial Y}\cdot\frac{\partial Y_{xyz\ldots}}{\partial x}+\cdots \quad Q.E.D.$$

SUPPLEMENT G.

NEW METHODS OF TRAJECTORY COMPUTATION.

Since this book went to press, a new method of trajectory computation and a new method of starting computations have been devised by a member of the Technical Staff. There is room now only for a bare outline of these methods, without going into the proof of the formulas involved.

TRAJECTORY COMPUTATION.

The method of trajectory computation may best be described by comparing it with that of Chapter VIII.

The first four or five lines are computed as laid down in that chapter, or by the method hereinafter described. Thereafter, the present method makes use of "anti-differences" of y'' and x'' These are written Δ^{-1} and Δ^{-2}. Thus, if we have a tabulation of Δ^{-2}, Δ^{-1}, y'', a, b, c, etc. (cf. p. 30), Δ^{-1} is the first difference of Δ^{-2}; y'' is the first difference of Δ^{-1}, and hence the second difference of Δ^{-2}, etc.

Now, since, for instance, 25 and 30, 93 and 98, 1000 and 1005, all have the same difference, and hence either set could equally well serve as Δ^{-1} to produce $y''=5$, it becomes necessary to determine, in some manner, the initial values of Δ^{-1} and Δ^{-2}. By inserting in formula 121 (*post*) the values of y', y'', and the a, b, and c of y'' for any given line, the value of Δ^{-1} for that line can easily be ascertained. Do this for two or three successive lines; and, if necessary, adjust the values of Δ^{-1} thus found, so that the already tabulated values of y'' will be *exactly* the first differences of Δ^{-1}.

Similarly get the values of Δ^{-2}, from formula 122 (*post*) for several successive lines, adjusting as before. Similarly get values of the Δ^{-1} and Δ^{-2} of x''.

From this point on, proceed practically as in Chapter VIII, except that we obtain x', x, y', and y, *directly* by the formulas of this supplement,[1] instead of $\Delta x'$, Δx, $\Delta y'$, and Δy, as in Chapter VIII. Integrate ahead by formulas 33 or 35 of that chapter. For the tentative value of y, to use in getting *log H*, we may disregard all terms of formula 34 except:

120 $$\int_{t-1}^{t} y'\,dt = i\,(y'_t - \tfrac{1}{2}a_t)$$

Compute the tentative values of y'' and x'' on the small sheet, as in Chapter VIII, and enter them on the trajectory sheet. Obtain the new Δ^{-1}, by algebraically adding the y'' of its line to the Δ^{-1} of the line before. Obtain the new Δ^{-2}, by algebraically adding the Δ^{-1} of its line to the Δ^{-2} of the line before. Similarly for x''.

[1] A Technical Staff paper, accompanying a Sample Trajectory, gives additional formulas which further lighten the labor of integration in special cases.

Then, instead of getting the *increment* of y' by formula 34 of Chapter VIII, get y' *itself* by:

$$121.[2] \qquad y'_t = i\left(\Delta_t^{-1} - \frac{1}{2}y''_t - \frac{1}{12}a_t - \frac{1}{24}b_t - \frac{1}{40}c_t\right)$$

and, instead of then integrating y' to get the *increment* of y, get y *itself direct* from y'' by:

$$122. \qquad y_t = i^2\left(\Delta^{-2} - \Delta^{-1} + \frac{1}{12}y''_t - \frac{b_t + c_t}{240}\right)$$

Note that the first difference, a, does not occur in this formula; and that i is squared. The first two terms in the parentheses can be consolidated into Δ^{-2}_{t-i}. The last term in the parentheses can usually be neglected or estimated.

x' can be found from formula 121 using x'' and its differences, and x (whenever needed) from formula 122, in each formula substituting x's for all the y's.

No columns of differences of y', y, x', or x need be carried on the trajectory sheet, unless desired as a check on the smoothness with which these functions are developing. Also, at least one less column of differences need be carried for y'' and x'' than in Chapter VIII. Thus, in place of the 25 columns of figures tabulated on the trajectory sheet in the method of that chapter, the present method can get along with 17 columns, as follows:

$$x,\ x',\ \Delta^{-2},\ \Delta^{-1},\ x'',\ a,\ b,\ c,\ t,\ y,\ y',\ \Delta^{-2},\ \Delta^{-1},\ y'',\ a,\ b,\ c.$$

The chief advantages of this new method are: It necessitates fewer columns of figures; its integration formulas are more rapidly convergent than those of Chapter VIII; fourth differences, which are apt to be very erratic in practice, are not employed in this method; any errors in integrating by this method are noncumulative, except of course as affecting x'' and y'' through the small-sheet computations, whereas in the method of Chapter VIII errors have this effect and in addition cumulate in their own columns. All that has to be paid for this gain is a slight artificiality of method, and the slight additional labor of starting the Δ^{-1} and Δ^{-2} columns.

[2] If a and c are running so smoothly that a_{t+i}, c_{t+i} and c_{t+2i} can be estimated, the following can be substituted for formula 121:

$$y' = \frac{i}{2}\left[\Delta^{-1}_{t-i} + \Delta^{-1}_{t} - \frac{1}{12}\left(a_t + a_{t+i}\right) + \frac{1}{60}\left(c_{t+i} + c_{t+2i}\right)\right]$$

This formula has obvious advantages, if the time-interval is two seconds.

STARTING THE TRAJECTORY.

The new method of starting trajectories is applicable either to the foregoing method or to that of Chapter VIII, except on *very* flat trajectories.

The values of the various elements for $t=0$ are computed as in Chapter VIII. Tentative values for log Ex' and log Ey' for $t=i$ are then extrapolated, by subtracting from the values for $t=0$ the respective quantities $DL\ Ex'$ and $DL\ Ey'$, found by the formulas:

123.
$$10^4\ DL\ Ex' = AE + By'$$
$$10^4\ DL\ Ey' = 10^4\ DL\ Ex' + C$$

If i is other than one second, multiply each of these by i before subtracting.

The factor 10^4 is included for convenience in connection with four place logarithms. The decrements found by formulas 123 represent the change in the fourth decimal place of the logarithm in question. The logarithms mentioned in this chapter are all denary, although natural logarithms were used in the derivation of the formulas.

Log A and log B are given on a one-page table ("Auxiliaries for Starting the Trajectory") with the argument $\frac{v^2}{100}$. C is from this table, by means of the argument y'. These symbols should not be confused with the three rotation coefficients of Chapter XIII, nor with the ballistic coefficient C.

The following is the explanation of the symbols involved:

$$DL\ Ex' = \frac{d}{dt} \log Ex'$$
$$A = 10^4\ Mn = M\left(1 + \frac{v}{G}\frac{dG}{dv}\right)10^4 \text{ (see p. 64)}$$
$$B = M\left(h + \frac{g}{vG}\frac{dG}{dv}\right)10^4$$
$$C = \left(\frac{Mg}{y'}\right)10^4$$
$$M = \log e = 0.434294 \quad . \quad . \quad .$$

Only a rough interpolation for log A and log B is necessary. For a y' less than 100, *multiply* y' by 10, enter the table with the result and *multiply* by 10 the C thus obtained.

The tentative values of log Ex' and log Ey', resulting from the use of the foregoing formulas, yield tentative values of x'' and y'', which can be integrated in the manner of Chapter VIII, to y', y, *and* x', with which to compute the column for $t=i$ on the small sheet as in that chapter.

Repeating the same process upon the final figures of the column for $t=i$, we obtain tentative values of y', y, and x for $t=2i$. And so on, until enough figures are tabulated on the trajectory sheet to supply sufficient differences for integration ahead. The use of this special method can then be discontinued, and the trajectory completed, either by the method of Chapter VIII or by the method discussed at the beginning of this supplement.

This special method of starting trajectories makes possible a very close approximation of x'' and y'', and thus obviates the repeated recomputation of the first few time intervals of the trajectory.[3]

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[3] It is claimed that this method will enable the computer to begin a trajectory with one-second intervals. There is some doubt as to the validity of this claim for very high velocities or very low ballistic coefficients. This method is to be used merely to yield *approximate* values of x'' and y'', on which to base the usual method of successive approximations. It is mathematically equivalent to obtaining correct values for x''' and y''' at $t=t_0$, and assuming that this value remains unaltered during the interval. It therefore materially reduces the number of approximations necessary to get a satisfactory value of x' and y'.

Supplement H.

NOTE ON ADVANCING DIFFERENCE FORMULAS.

The student's attention is directed to the fact that, in most mathematical books that treat of interpolation and integration, *advancing* differences are used almost exclusively, instead of the *receding* differences used throughout this book. Therefore, the formulas of most textbooks on finite differences will not be those of this book.

The computations of ballistics that involve interpolation or integration are mostly of such a nature that the work continually lies at or very near the *last* one of those values of the functions already derived and tabulated. (In the formula for "integration ahead," for instance, one limit of our integral, and, in fact, the entire integration interval, lies *beyond* the time of our latest established and tabulated value of the function.) Hence, we are continually so situated that the only differences available to us in this region are receding differences. We are, therefore, unable to use the advancing difference formulas usually given in textbooks.

The student who wishes to construct advancing difference formulas, for use in special cases, may use either of the methods given in this note. It is, however, advised that all such formulas be plainly labeled to indicate that they are to be used *only* with *advancing* differences.

Advancing differences are formed thus:

$$a_0 = f_1 - f_0 \text{ (instead of } f_0 - f_{-1}),$$
$$a_1 = f_2 - f_1 \text{ (instead of } f_1 - f_0), \text{ etc.}$$
$$b_0 = a_1 - a_0 \text{ (instead of } a_0 - a_{-1}), \text{ etc.}$$

That is, $f_2 - f_1$ is now called a_1 (instead of a_2); $f_2 - 2f_1 + f_0$, which, being the old $a_2 - a_1$, was called b_2, has now become $a_1 - a_0 = b_0$; etc.

It is to be noted that, whether receding or advancing differences are used, the subtractions are always made with same direction; i. e., we always have $f_2 - f_1$, $a_3 - a_2$, etc., never $f_1 - f_2$ or $a_2 - a_3$.

One method of obtaining any advancing difference formula from the corresponding receding difference formula (see footnotes, pages 17 and 31) is as follows:

In the formula for receding differences, change the algebraic sign of every *odd*-ordered difference, the even-ordered remaining unchanged. Also, change the sign of every t (and therefore of every Δt) throughout the equation.

The last condition requires that any integral shall have the sign of dt and also of both the limits changed. Thus the formula:

$$+\int_{+1}^{+2} f dt = \text{terms in } f, a, b, c, \text{ etc. (receding)},$$

will become

$$+\int_{-1}^{-2} f(-dt) = -\int_{-1}^{-2} f dt = +\int_{-2}^{-1} f dt = \text{terms in } f,\ a,\ b,\ c, \text{ etc. (advancing)};$$

the coefficients of f, a, b, c, etc., in one equation, being respectively equal to the coefficients of f, a, b, c, etc., in the other equation, but in the second equation the coefficients of a, c, etc. (the *odd*-ordered differences), are each opposite in sign to the same coefficients in the other equation. Referring to the schedule of page 30, it is noted that if the first equation above uses f_0, a_0, b_0, c_0, etc., of the *receding* difference notation, then the second equation will use f_0, a_0, b_0, c_0, etc., of the *advancing* difference notation, which will be respectively equal to f_0, a_1, b_2, c_3, etc., of the *receding* difference notation of the schedule.

The preceding process is reversible: i. e., exactly the same changes are to be made in transforming any formula for *advancing* differences into the corresponding formula for *receding* differences.

A full set of advancing difference formulas can be derived *ab initio*, by tabulating f_0, f_1, f_2, etc., and the advancing differences; and then performing the work of problems 28 to 35, and 39 and 41; noting that the algebraic signs of all subscripts, limits, and values of t in general, in the statements of those problems, are to be changed.

The expression: "*corresponding* formula," which has been used in the preceding discussion, needs some explanation. If a formula uses differences *receding* (or advancing) from f_0, to interpolate for f_n, the "corresponding" formula in differences *advancing* (or receding) from f_0 will interpolate for f_{-n}. If a formula using differences *receding* (or advancing) from f_0 gives the value of $\int_m^n f dt$, the "corresponding" formula in differences *advancing* (or receding) from f_0, will give $-\int_{-m}^{-n} f dt$. For example, the formula in *receding* differences, for "integration ahead" from f_0 transforms into a formula using *advancing* differences for integrating over the interval *back* from f_0.

It should be noted that even in Chapter XIV, where the function to be integrated is tabulated in reverse order of time, yet, since the differences used recede with respect to the order of *tabulation*, only receding difference formulas are to be employed. (See footnote, p. 30.)

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